3.4 Isomorphisms

A one-to-one correspondence $\phi : G_1 \to G_2$ between groups G_1 and G_2 is called a group isomorphism if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G_1$. The function ϕ can be thought of as simply renaming the elements of G_1, since it is one-to-one and onto. The condition that $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G_1$ makes certain that multiplication can be done in either group and the transferred to the other, since the inverse function ϕ^{-1} also respects the multiplication of the two groups.

In terms of the respective group multiplication tables for G_1 and G_2, the existence of an isomorphism guarantees that there is a way to set up a correspondence between the elements of the groups in such a way that the group multiplication tables will look exactly the same.

From an algebraic perspective, we should think of isomorphic groups as being essentially the same. The problem of finding all abelian groups of order 8 is impossible to solve, because there are infinitely many possibilities. But if we ask for a list of abelian groups of order 8 that comes with a guarantee that *any* possible abelian group of order 8 must be isomorphic to one of the groups on the list, then the question becomes manageable. In fact, we can show (in Section 7.5) that the answer to this particular question is the list $\mathbf{Z}_8$, $\mathbf{Z}_4 \times \mathbf{Z}_2$, $\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. In this situation we would usually say that we have found all abelian groups of order 8, *up to isomorphism*.

To show that two groups G_1 and G_2 are isomorphic, you should actually produce an isomorphism $\phi : G_1 \to G_2$. To decide on the function to use, you probably need to see some similarity between the group operations.

In some ways it is harder to show that two groups are *not* isomorphic. If you can show that one group has a property that the other one does not have, then you can decide that two groups are not isomorphic (provided that the property would have been transferred by any isomorphism). Suppose that G_1 and G_2 are isomorphic groups. If G_1 is abelian, then so is G_2; if G_1 is cyclic, then so is G_2. Furthermore, for each positive integer n, the two groups must have exactly the same number of elements of order n. Each time you meet a new property of groups, you should ask whether it is preserved by any isomorphism.

SOLVED PROBLEMS: §3.4

21. Show that $\mathbf{Z}_{17}^{\times}$ is isomorphic to $\mathbf{Z}_{16}$.

22. Let $\phi : \mathbf{R}^{\times} \to \mathbf{R}^{\times}$ be defined by $\phi(x) = x^3$, for all $x \in \mathbf{R}$. Show that ϕ is a group isomorphism.

23. Let G_1, G_2, H_1, H_2 be groups, and suppose that $\theta_1 : G_1 \to H_1$ and $\theta_2 : G_2 \to H_2$ are group isomorphisms. Define $\phi : G_1 \times G_2 \to H_1 \times H_2$ by $\phi(x_1, x_2) = (\theta_1(x_1), \theta_2(x_2))$, for all $(x_1, x_2) \in G_1 \times G_2$. Prove that ϕ is a group isomorphism.

24. Prove that the group $\mathbf{Z}_7^\times \times \mathbf{Z}_{11}^\times$ is isomorphic to the group $\mathbf{Z}_6 \times \mathbf{Z}_{10}$.

25. Define $\phi : \mathbf{Z}_{30} \times \mathbf{Z}_2 \to \mathbf{Z}_{10} \times \mathbf{Z}_6$ by $\phi([n]_{30}, [m]_2) = ([n]_{10}, [4n+3m]_6)$, for all $([n]_{30}, [m]_2) \in \mathbf{Z}_{30} \times \mathbf{Z}_2$. First prove that ϕ is a well-defined function, and then prove that ϕ is a group isomorphism.

26. Let G be a group, and let H be a subgroup of G. Prove that if a is any element of G, then the subset

$$aHa^{-1} = \{g \in G \mid g = aha^{-1} \text{ for some } h \in H\}$$

is a subgroup of G that is isomorphic to H.

27. Let G, G_1, G_2 be groups. Prove that if G is isomorphic to $G_1 \times G_2$, then there are subgroups H and K in G such that $H \cap K = \{e\}$, $HK = G$, and $hk = kh$ for all $h \in H$ and $k \in K$.

28. Show that for any prime number p, the subgroup of diagonal matrices in $\mathrm{GL}_2(\mathbf{Z}_p)$ is isomorphic to $\mathbf{Z}_p^\times \times \mathbf{Z}_p^\times$.

29. (a) In the group $G = GL_2(\mathbf{R})$ of invertible 2×2 matrices with real entries, show that

$$H = \left\{ \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right] \in GL_2(\mathbf{R}) \middle| \, a_{11} = 1,\ a_{21} = 0,\ a_{22} = 1 \right\}$$

is a subgroup of G.

(b) Show that H is isomorphic to the group $\mathbf{R}$ of all real numbers, under addition.

30. Let G be the subgroup of $\mathrm{GL}_2(\mathbf{R})$ defined by

$$G = \left\{ \left[\begin{array}{cc} m & b \\ 0 & 1 \end{array} \right] \middle| \, m \neq 0 \right\} .$$

Show that G is not isomorphic to the direct product $\mathbf{R}^\times \times \mathbf{R}$.

31. Let H be the following subgroup of group $G = GL_2(\mathbf{Z}_3)$.

$$H = \left\{ \left[\begin{array}{cc} m & b \\ 0 & 1 \end{array} \right] \in GL_2(\mathbf{Z}_3) \middle| \, m, b \in \mathbf{Z}_3,\ m \neq 0 \right\}$$

Show that H is isomorphic to the symmetric group $\mathcal{S}_3$.

32. Let G be a group, and let S be any set for which there exists a one-to-one and onto function $\phi : G \to S$. Define an operation on S by setting $x_1 \cdot x_2 = \phi(\phi^{-1}(x_1)\phi^{-1}(x_2))$, for all $x_1, x_2 \in S$. Prove that S is a group under this operation, and that ϕ is actually a group isomorphism.