3.5 Cyclic Groups

We began our study of abstract algebra very concretely, by looking at the group $\mathbf{Z}$ of integers, and the related groups $\mathbf{Z}_n$. We discovered that each of these groups is generated by a single element, and this motivated the definition of an abstract cyclic group. In this section, Theorem 3.5.2 shows that every cyclic group is isomorphic to one of these concrete examples, so all of the information about cyclic groups is already contained in these basic examples.

You should pay particular attention to Proposition 3.5.3, which describes the subgroups of $\mathbf{Z}_n$, showing that they are in one-to-one correspondence with the positive divisors of n. In n is a prime power, then the subgroups are "linearly ordered" in the sense that given any two subgroups, one is a subset of the other. These cyclic groups have a particularly simple structure, and form the basic building blocks for *all* finite abelian groups. (In Theorem 7.5.4 we will prove that every finite abelian group is isomorphic to a direct product of cyclic groups of prime power order.)

SOLVED PROBLEMS: §3.5

20. Show that the three groups $\mathbf{Z}_6$, $\mathbf{Z}_9^\times$, and $\mathbf{Z}_{18}^\times$ are isomorphic to each other.

21. Is $\mathbf{Z}_4 \times \mathbf{Z}_{10}$ isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_{20}$?

22. Is $\mathbf{Z}_4 \times \mathbf{Z}_{15}$ isomorphic to $\mathbf{Z}_6 \times \mathbf{Z}_{10}$?

23. Give the lattice diagram of subgroups of $\mathbf{Z}_{100}$.

24. Find all generators of the cyclic group $\mathbf{Z}_{28}$.

25. In $\mathbf{Z}_{30}$, find the order of the subgroup $\langle [18]_{30} \rangle$; find the order of $\langle [24]_{30} \rangle$.

26. Prove that if G_1 and G_2 are groups of order 7 and 11, respectively, then the direct product $G_1 \times G_2$ is a cyclic group.

27. Show that any cyclic group of even order has exactly one element of order 2.

28. Use the the result in Problem 27 to show that the multiplicative groups $\mathbf{Z}_{15}^\times$ and $\mathbf{Z}_{21}^\times$ are not cyclic groups.

29. Find all cyclic subgroups of the quaternion group. Use this information to show that the quaternion group cannot be isomorphic to the subgroup of $\mathcal{S}_4$ generated by $(1, 2, 3, 4)$ and $(1, 3)$.

30. Prove that if p and q are different odd primes, then $\mathbf{Z}_{pq}^\times$ is not a cyclic group.