3.6 Permutation Groups

As with the previous section, this section revisits the roots of group theory that we began to study in an earlier chapter. Cayley's theorem shows that permutation groups contain all of the information about finite groups, since every finite group of order n is isomorphic to a subgroup of the symmetric group $\mathcal{S}_n$. That isn't as impressive as it sounds at first, because as n gets larger and larger, the subgroups of order n just get lost inside the larger symmetric group, which has order $n!$. This does imply, however, that from the algebraists point of view the abstract definition of a group is really no more general than the concrete definition of a permutation group. The abstract definition of a group is useful simply because it can be more easily applied to a wide variety of situation.

You should make every effort to get to know the dihedral groups $\mathcal{D}_n$. They have a concrete representation, in terms of the rigid motions of an n-gon, but can also be described more abstractly in terms of two generators a (of order n) and b (of order 2) which satisfy the relation $ba = a^{-1}b$. We can write

$$\mathcal{D}_n = \{a^i b^j \mid 0 \le i < n,\ 0 \le j < 2,\ \text{with } o(a) = n,\ o(b) = 2,\ \text{and } ba = a^{-1}b\} \ .$$

In doing computations in $\mathcal{D}_n$ it is useful to have at hand the formula $ba^i = a^{n-i}b$, shown in the first of the solved problems given below.

SOLVED PROBLEMS: §3.6

22. In the dihedral group $\mathcal{D}_n = \{a^i b^j \mid 0 \le i < n,\ 0 \le j < 2\}$ with $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$, show that $ba^i = a^{n-i}b$, for all $0 \le i < n$.

23. In the dihedral group $\mathcal{D}_n = \{a^i b^j \mid 0 \le i < n,\ 0 \le j < 2\}$ with $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$, show that each element of the form $a^i b$ has order 2.

24. In $\mathcal{S}_4$, find the subgroup H generated by $(1,2,3)$ and $(1,2)$.

25. For the subgroup H of $\mathcal{S}_4$ defined in the previous problem, find the corresponding subgroup $\sigma H \sigma^{-1}$, for $\sigma = (1,4)$.

26. Show that each element in $\mathcal{A}_4$ can be written as a product of 3-cycles.

27. In the dihedral group $\mathcal{D}_n = \{a^i b^j \mid 0 \le i < n,\ 0 \le j < 2\}$ with $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$, find the centralizer of a.

28. Find the centralizer of $(1,2,3)$ in $\mathcal{S}_3$, in $\mathcal{S}_4$, and in $\mathcal{A}_4$.