3.7 Homomorphisms

In Section 3.4 we introduced the concept of an isomorphism, and studied in detail what it means for two groups to be isomorphic. In this section we look at functions that respect the group operations but may not be one-to-one and onto. There are many important examples of group homomorphisms that are not isomorphisms, and, in fact, homomorphisms provide the way to relate one group to another.

The most important result in this section is Theorem 3.7.8, which is a preliminary form of the Fundamental Homomorphism Theorem. (The full statement is given in Theorem 3.8.8, after we develop the concepts of cosets and factor groups.) In this formulation of the Fundamental Homomorphism Theorem, we start with a group homomorphism $\phi : G_1 \to G_2$. It is easy to prove that the image $\phi(G_1)$ is a subgroup of G_2. The function ϕ has an equivalence relation associated with it, where we let $a \sim b$ if $\phi(a) = \phi(b)$, for $a, b \in G_1$. Just as in $\mathbf{Z}$, where we use the equivalence relation defined by congruence modulo n, we can define a group operation on the equivalence classes of $\sim$, using the operation in G_1. Then Theorem 3.7.8 shows that this group is isomorphic to $\phi(G_1)$, so that although the homomorphism may not be an isomorphism between G_1 and G_2, it *does* define an isomorphism between a subgroup of G_2 and what we call a *factor group* of G_1.

Proposition 3.7.6 is also useful, since for any group homomorphism $\phi : G_1 \to G_2$ it describes the connections between subgroups of G_1 and subgroups of G_2. Examples 3.7.4 and 3.7.5 are important, because they give a complete description of all group homomorphisms between two cyclic groups.

SOLVED PROBLEMS: §3.7

17. Find all group homomorphisms from $\mathbf{Z}_4$ into $\mathbf{Z}_{10}$.

18. (a) Find the formulas for all group homomorphisms from $\mathbf{Z}_{18}$ into $\mathbf{Z}_{30}$.

    (b) Choose one of the nonzero formulas in part (a), and for this formula find the kernel and image, and show how elements of the image correspond to cosets of the kernel.

19. (a) Show that $\mathbf{Z}_7^\times$ is cyclic, with generator $[3]_7$.

    (b) Show that $\mathbf{Z}_{17}^\times$ is cyclic, with generator $[3]_{17}$.

    (c) Completely determine all group homomorphisms from $\mathbf{Z}_{17}^\times$ into $\mathbf{Z}_7^\times$.

20. Define $\phi : \mathbf{Z}_4 \times \mathbf{Z}_6 \to \mathbf{Z}_4 \times \mathbf{Z}_3$ by $\phi(x, y) = (x + 2y, y)$.

    (a) Show that ϕ is a well-defined group homomorphism.

    (b) Find the kernel and image of ϕ, and apply the fundamental homomorphism theorem.

21. Let n and m be positive integers, such that m is a divisor of n. Show that $\phi : \mathbf{Z}_n^\times \to \mathbf{Z}_m^\times$ defined by $\phi([x]_n) = [x]_m$, for all $[x]_n \in \mathbf{Z}_n^\times$, is a well-defined group homomorphism.

22. For the group homomorphism $\phi : \mathbf{Z}_{36}^\times \to \mathbf{Z}_{12}^\times$ defined by $\phi([x]_{36}) = [x]_{12}$, for all $[x]_{36} \in \mathbf{Z}_{36}^\times$, find the kernel and image of ϕ, and apply the fundamental homomorphism theorem.

23. Let G, G_1, and G_2 be groups. Let $\phi_1 : G \to G_1$ and $\phi_2 : G \to G_2$ be group homomorphisms. Prove that $\phi : G \to G_1 \times G_2$ defined by $\phi(x) = (\phi_1(x), \phi_2(x))$, for all $x \in G$, is a well-defined group homomorphism.

24. Let p and q be different odd primes. Prove that $\mathbf{Z}_{pq}^\times$ is isomorphic to the direct product $\mathbf{Z}_p^\times \times \mathbf{Z}_q^\times$.