3.8 Cosets, Normal Subgroups, and Factor Groups

The notion of a factor group is one of the most important concepts in abstract algebra. To construct a factor group, we start with a normal subgroup and the equivalence classes it determines. This construction parallels the construction of $\mathbf{Z}_n$ from $\mathbf{Z}$, where we have $a \equiv b \pmod{n}$ if and only if $a - b \in n\mathbf{Z}$. The only complication is that the equivalence relation respects the operation in G only when the subgroup is a normal subgroup. Of course, in an abelian group we can use any subgroup, since all subgroups of an abelian group are normal.

The key idea is to begin thinking of equivalence classes as elements in their own right. That is what we did in Chapter 1, where at first we thought of congruence classes as infinite sets of integers, and then in Section 1.4 when we started working with $\mathbf{Z}_n$ we started to use the notation $[a]_n$ to suggest that we were now thinking of a single element of a set.

In actually using the Fundamental Homomorphism Theorem, it is important to let the theorem do its job, so that it does as much of the hard work as possible. Quite often we need to show that a factor group G/N that we have constructed is isomorphic to another group G_1. The easiest way to do this is to just define a homomorphism ϕ from G to G_1, making sure that N is the kernel of ϕ. If you prove that ϕ maps G onto G_1, then the Fundamental Theorem does the rest of the work, showing that there exists a well-defined isomorphism between G/N and G_1.

The moral of this story is that if you define a function on G rather than G/N, you ordinarily don't need to worry that it is well-defined. On the other hand, if you define a function on the cosets of G/N, the most convenient way is use a formula defined on representatives of the cosets of N. But then you must be careful to prove that the formula you are using does not depend on the particular choice of a representative. That is, you must prove that your formula actually defines a function. Then you must prove that your function is one-to-one, in addition to proving that it is onto and respects the operations in the two groups. Once again, if your function is defined on cosets, it can be much trickier to prove that it is one-to-one than to simply compute the kernel of a homomorphism defined on G.

SOLVED PROBLEMS: §3.8

27. List the cosets of $\langle 7 \rangle$ in $\mathbf{Z}_{16}^{\times}$. Is the factor group $\mathbf{Z}_{16}^{\times} / \langle 7 \rangle$ cyclic?

28. Let $G = \mathbf{Z}_6 \times \mathbf{Z}_4$, let $H = \{(0,0),(0,2)\}$, and let $K = \{(0,0),(3,0)\}$.

    (a) List all cosets of H; list all cosets of K.

    (b) You may assume that any abelian group of order 12 is isomorphic to either $\mathbf{Z}_{12}$ or $\mathbf{Z}_6 \times \mathbf{Z}_2$. Which answer is correct for G/H? For G/K?

29. Let the dihedral group D_n be given via generators and relations, with generators a of order n and b of order 2, satisfying $ba = a^{-1}b$.

(a) Show that $ba^i = a^{-i}b$ for all i with $1 \leq i < n$.

(b) Show that any element of the form $a^i b$ has order 2.

(c) List all left cosets and all right cosets of $\langle b \rangle$

30. Let $G = D_6$ and let N be the subgroup $\left\langle a^3 \right\rangle = \{e, a^3\}$ of G.

    (a) Show that N is a normal subgroup of G.

    (b) Is G/N abelian?

31. Let G be the dihedral group D_{12}, and let $N = \{e, a^3, a^6, a^9\}$.

    (a) Prove that N is a normal subgroup of G, and list all cosets of N.

    (b) You may assume that G/N is isomorphic to either $\mathbf{Z}_6$ or S_3. Which is correct?

32. (a) Let G be a group. For $a, b \in G$ we say that b is conjugate to a, written $b \sim a$, if there exists $g \in G$ such that $b = gag^{-1}$. Show that $\sim$ is an equivalence relation on G. The equivalence classes of $\sim$ are called the *conjugacy classes* of G.

    (b) Show that a subgroup N of G is normal in G if and only if N is a union of conjugacy classes.

33. Find the conjugacy classes of D_4.

34. Let G be a group, and let N and H be subgroups of G such that N is normal in G.

    (a) Prove that HN is a subgroup of G.

    (b) Prove that N is a normal subgroup of HN.

    (c) Prove that if $H \cap N = \{e\}$, then HN/N is isomorphic to H.