Review Problems

1. (a) What are the possibilities for the order of an element of $\mathbf{Z}_{13}^{\times}$? Explain your answer.

   (b) Show that $\mathbf{Z}_{13}^{\times}$ is a cyclic group.

2. Find all subgroups of $\mathbf{Z}_{11}^{\times}$, and give the lattice diagram which shows the inclusions between them.

3. Let G be the subgroup of $GL_3(\mathbf{R})$ consisting of all matrices of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{such that } a, b \in \mathbf{R}\,.$$

   Show that G is a subgroup of $GL_3(\mathbf{R})$.

4. Show that the group G in the previous problem is isomorphic to the direct product $\mathbf{R} \times \mathbf{R}$.

5. List the cosets of the cyclic subgroup $\langle 9 \rangle$ in $\mathbf{Z}_{20}^{\times}$. Is $\mathbf{Z}_{20}^{\times} / \langle 9 \rangle$ cyclic?

6. Let G be the subgroup of $GL_2(\mathbf{R})$ consisting of all matrices of the form $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$, and let N be the subset of all matrices of the form $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$.

   (a) Show that N is a subgroup of G, and that N is normal in G.

   (b) Show that G/N is isomorphic to the multiplicative group $\mathbf{R}^{\times}$.

7. Assume that the dihedral group D_4 is given as $\{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, where $a^4 = e$, $b^2 = e$, and $ba = a^3b$. Let N be the subgroup $\langle a^2 \rangle = \{e, a^2\}$.

   (a) Show by a direct computation that N is a normal subgroup of D_4.

   (b) Is the factor group D_4/N a cyclic group?

8. Let $G = D_8$, and let $N = \{e, a^2, a^4, a^6\}$.

   (a) List all left cosets and all right cosets of N, and verify that N is a normal subgroup of G.

   (b) Show that G/N has order 4, but is not cyclic.