4

POLYNOMIALS

In this chapter we return to several of the themes in Chapter 1. We need to talk about the greatest common divisor of two polynomials, and when two polynomials are relatively prime. The notion of a prime number is replaced by that of an *irreducible polynomial*. We can work with congruence classes of polynomials, just as we did with congruence classes of integers. The point of saying this is that it will be worth your time to review the definitions and theorems in Chapter 1.

In addition to generalizing ideas from the integers to polynomials, we want to go beyond high school algebra, to be able to work with coefficients that may not be real numbers. This motivates the definition of a field, which is quite closely related to the definition of a group (now there are two operations instead of just one). The point here is that you can benefit from reviewing Chapter 3.

Because you have a lot more experience now than when you started Chapter 1, I didn't break the problems up by section. Of course, you don't have to wait until you have finished the chapter to practice solving some of these problems.

Review Problems

1. Use the Euclidean algorithm to find $\gcd(x^8 - 1,\ x^6 - 1)$ in $\mathbf{Q}[x]$ and write it as a linear combination of $x^8 - 1$ and $x^6 - 1$.

2. Over the field of rational numbers, use the Euclidean algorithm to show that $2x^3 - 2x^2 - 3x + 1$ and $2x^2 - x - 2$ are relatively prime.

3. Over the field of rational numbers, find the greatest common divisor of $x^4 + x^3 + 2x^2 + x + 1$ and $x^3 - 1$, and express it as a linear combination of the given polynomials.

4. Over the field of rational numbers, find the greatest common divisor of $2x^4 - x^3 + x^2 + 3x + 1$ and $2x^3 - 3x^2 + 2x + 2$ and express it as a linear combination of the given polynomials.

5. Are the following polynomials irreducible over $\mathbf{Q}$?

   (a) $3x^5 + 18x^2 + 24x + 6$

   (b) $7x^3 + 12x^2 + 3x + 45$

   (c) $2x^{10} + 25x^3 + 10x^2 - 30$

6. Factor $x^5 - 10x^4 + 24x^3 + 9x^2 - 33x - 12$ over $\mathbf{Q}$.

7. Factor $x^5 - 2x^4 - 2x^3 + 12x^2 - 15x - 2$ over $\mathbf{Q}$.

8. (a) Show that $x^2 + 1$ is irreducible over $\mathbf{Z}_3$.

   (b) List the elements of the field $F = \mathbf{Z}_3[x]/\left\langle x^2 + 1 \right\rangle$.

   (c) In the multiplicative group of nonzero elements of F, show that $[x + 1]$ is a generator, but $[x]$ is not.

9. (a) Express $x^4 + x$ as a product of polynomials irreducible over $\mathbf{Z}_5$.

   (b) Show that $x^3 + 2x^2 + 3$ is irreducible over $\mathbf{Z}_5$.

10. Express $2x^3 + x^2 + 2x + 2$ as a product of polynomials irreducible over $\mathbf{Z}_5$.

11. Construct an example of a field with $343 = 7^3$ elements.

12. In $\mathbf{Z}_2[x]/\left\langle x^3 + x + 1 \right\rangle$, find the multiplicative inverse of $[x + 1]$.

13. Find the multiplicative inverse of $[x^2 + x + 1]$

    (a) in $\mathbf{Q}[x]/\left\langle x^3 - 2 \right\rangle$;

    (b) in $\mathbf{Z}_3[x]/\left\langle x^3 + 2x^2 + x + 1 \right\rangle$.

14. In $\mathbf{Z}_5[x]/\left\langle x^3 + x + 1 \right\rangle$, find $[x]^{-1}$ and $[x + 1]^{-1}$, and use your answers to find $[x^2 + x]^{-1}$.

15. Factor $x^4 + x + 1$ over $\mathbf{Z}_2[x]/\left\langle x^4 + x + 1 \right\rangle$.