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COMMUTATIVE RINGS

This chapter takes its motivation from Chapter 1 and Chapter 4, extending results on factorization to more general settings than just the integers or polynomials over a field. The concept of a factor ring depends heavily on the corresponding definition for groups, so you may need to review the last two sections of Chapter 3. Remember that the distributive law is all that connects the two operations in a ring, so it is crucial in many of the proofs you will see.

Review Problems

1. Let R be the ring with 8 elements consisting of all 3×3 matrices with entries in $\mathbf{Z}_2$ which have the following form:

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}$$

You may assume that the standard laws for addition and multiplication of matrices are valid.

(a) Show that R is a commutative ring (you only need to check closure and commutativity of multiplication).

(b) Find all units of R, and all nilpotent elements of R.

(c) Find all idempotent elements of R.

2. Let R be the ring $\mathbf{Z}_2[x]/\langle x^2+1\rangle$. Show that although R has 4 elements, it is not isomorphic to either of the rings $\mathbf{Z}_4$ or $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

3. Find all ring homomorphisms from $\mathbf{Z}_{120}$ into $\mathbf{Z}_{42}$.

4. Are $\mathbf{Z}_9$ and $\mathbf{Z}_3 \oplus \mathbf{Z}_3$ isomorphic as rings?

5. In the group $\mathbf{Z}_{180}^{\times}$ of units of the ring $\mathbf{Z}_{180}$, what is the largest possible order of an element?

6. For the element $a = (0,2)$ of the ring $R = \mathbf{Z}_{12} \oplus \mathbf{Z}_8$, find $\text{Ann}(a) = \{r \in R \mid ra = 0\}$. Show that $\text{Ann}(a)$ is an ideal of R.

7. Let R be the ring $\mathbf{Z}_2[x]/\langle x^4+1\rangle$, and let I be the set of all congruence classes in R of the form $[f(x)(x^2+1)]$.

   (a) Show that I is an ideal of R.

   (b) Show that $R/I \cong \mathbf{Z}_2[x]/\langle x^2+1\rangle$.

   (c) Is I a prime ideal of R?

   Hint: If you use the fundamental homomorphism theorem, you can do the first two parts together.

8. Find all maximal ideals, and all prime ideals, of $\mathbf{Z}_{36} = \mathbf{Z}/36\mathbf{Z}$.

9. Give an example to show that the set of all zero divisors of a ring need not be an ideal of the ring.

10. Let I be the subset of $\mathbf{Z}[x]$ consisting of all polynomials with even coefficients. Prove that I is a prime ideal; prove that I is not maximal.

11. Let R be any commutative ring with identity 1.

    (a) Show that if e is an idempotent element of R, then $1-e$ is also idempotent.

    (b) Show that if e is idempotent, then $R \cong Re \oplus R(1-e)$.

12. Let R be the ring $\mathbf{Z}_2[x]/\langle x^3+1\rangle$.

    (a) Find all ideals of R.

    (b) Find the units of R.

    (c) Find the idempotent elements of R.

13. Let S be the ring $\mathbf{Z}_2[x]/\langle x^3+x\rangle$.

    (a) Find all ideals of S.

    (b) Find the units of R.

    (c) Find the idempotent elements of R.

14. Show that the rings R and S in the two previous problems are isomorphic as abelian groups, but not as rings.

15. Let $\mathbf{Z}[i]$ be the subring of the field of complex numbers given by

$$\mathbf{Z}[i] = \{m + ni \in \mathbf{C} \mid m, n \in \mathbf{Z}\} .$$

(a) Define $\phi : \mathbf{Z}[i] \to \mathbf{Z}_2$ by $\phi(m + ni) = [m + n]_2$. Prove that ϕ is a ring homomorphism. Find $\ker(\phi)$ and show that it is a principal ideal of $\mathbf{Z}[i]$.

(b) For any prime number p, define $\theta : \mathbf{Z}[i] \to \mathbf{Z}_p[x]/\left\langle x^2 + 1\right\rangle$ by $\theta(m+ni) = [m + nx]$. Prove that θ is an onto ring homomorphism.

16. Let I and J be ideals in the commutative ring R, and define the function $\phi : R \to R/I \oplus R/J$ by $\phi(r) = (r + I, r + J)$, for all $r \in R$.

(a) Show that ϕ is a ring homomorphism, with $\ker(\phi) = I \cap J$.

(b) Show that if $I + J = R$, then ϕ is onto, and thus $R/(I \cap J) \cong R/I \oplus R/J$.

17. Considering $\mathbf{Z}[x]$ to be a subring of $\mathbf{Q}[x]$, show that these two integral domains have the same quotient field.

18. Let p be an odd prime number that is not congruent to 1 modulo 4. Prove that the ring $\mathbf{Z}_p[x]/\left\langle x^2 + 1\right\rangle$ is a field.

Hint: Show that a root of $x^2 = -1$ leads to an element of order 4 in the multiplicative group $\mathbf{Z}_p^\times$.