

## 1.3 SOLUTIONS

26. Solve the congruence
- $42x \equiv 12 \pmod{90}$
- .

*Solution:* We have  $\gcd(42, 90) = 6$ , so there is a solution since 6 is a factor of 12. Solving the congruence  $42x \equiv 12 \pmod{90}$  is equivalent solving the equation  $42x = 12 + 90q$  for integers  $x$  and  $q$ . This reduces to  $7x = 2 + 15q$ , or  $7x \equiv 2 \pmod{15}$ . Equivalently, we obtain  $7x \equiv 2 \pmod{15}$  by dividing  $42x \equiv 12 \pmod{90}$  through by 6. We next use trial and error to look for the multiplicative inverse of 7 modulo 15. The numbers congruent to 1 modulo 15 are 16, 31, 46, 61, etc., and  $-14, -29, -34$ , etc. Among these, we see that 7 is a factor of  $-14$ , so we multiply both sides of the congruence by  $-2$  since  $(-2)(7) = -14 \equiv 1 \pmod{15}$ . Thus we have  $-14x \equiv -4 \pmod{15}$ , or  $x \equiv 11 \pmod{15}$ . The solution is  $x \equiv 11, 26, 41, 56, 71, 86 \pmod{90}$ .

27. (a) Find all solutions to the congruence
- $55x \equiv 35 \pmod{75}$
- .

*Solution:* We have  $\gcd(55, 75) = 5$ , which is a divisor of 35. Thus we have

$$55x \equiv 35 \pmod{75}; \quad 11x \equiv 7 \pmod{15}; \quad 44x \equiv 28 \pmod{15};$$

$$-x \equiv 13 \pmod{15}; \quad x \equiv 2 \pmod{15}. \quad \text{The solution is}$$

$$x \equiv 2, 17, 32, 47, 62 \pmod{75}.$$

- (b) Find all solutions to the congruence
- $55x \equiv 36 \pmod{75}$
- .

*Solution:* There is no solution, since  $\gcd(55, 75) = 5$  is not a divisor of 36.

28. (a) Find one particular integer solution to the equation
- $110x + 75y = 45$
- .

*Solution:* Any linear combination of 110 and 75 is a multiple of the gcd.

$$\begin{bmatrix} 1 & 0 & 110 \\ 0 & 1 & 75 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 35 \\ 0 & 1 & 75 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 35 \\ -2 & 3 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 15 & -22 & 0 \\ -2 & 3 & 5 \end{bmatrix}$$

Thus  $-2(110) + 3(75) = 5$ , and multiplying by 9 yields a solution  $x = -18$ ,  $y = 27$ .

*Comment:* The matrix computation shows that  $110(15) + 75(-22) = 0$ , so adding any multiple of the vector  $(15, -22)$  to the particular solution  $(-18, 27)$  will also determine a solution.

*Second solution:* The equation reduces to the congruence  $35x \equiv 45 \pmod{75}$ . This reduces to  $7x \equiv 9 \pmod{15}$ , and multiplying both sides by  $-2$  gives  $x \equiv -3 \pmod{15}$ . Thus  $75y = 45 + 3(110) = 375$  and so  $x = -3$ ,  $y = 5$  is a solution.

- (b) Show that if
- $x = m$
- and
- $y = n$
- is an integer solution to the equation in part (a), then so is
- $x = m + 15q$
- and
- $y = n - 22q$
- , for any integer
- $q$
- .

*Solution:* If  $110m + 75n = 45$ , then  $110(m + 15q) + 75(n - 22q) = 45 + 110(15)q + 75(-22)q = 45$ , since  $110(15) - 75(22) = 0$ .

29. Solve the system of congruences  $x \equiv 2 \pmod{9}$   $x \equiv 4 \pmod{10}$ .

*Solution:* Convert the second congruence to the equation  $x = 4 + 10q$  for some  $q \in \mathbf{Z}$ . Then  $4 + 10q \equiv 2 \pmod{9}$ , which reduces to  $q \equiv 7 \pmod{9}$ . Thus the solution is  $x \equiv 74 \pmod{90}$ .

30. Solve the system of congruences  $5x \equiv 14 \pmod{17}$   $3x \equiv 2 \pmod{13}$ .

*Solution:* By trial and error,  $7 \cdot 5 \equiv 1 \pmod{17}$  and  $9 \cdot 3 \equiv 1 \pmod{13}$ ,  
so  $5x \equiv 14 \pmod{17}$ ;  $35x \equiv 98 \pmod{17}$ ;  $x \equiv 13 \pmod{17}$   
and  $3x \equiv 2 \pmod{13}$ ;  $27x \equiv 18 \pmod{13}$ ;  $x \equiv 5 \pmod{13}$ .

Having reduced the system to the standard form, we can solve it in the usual way. We have  $x = 13 + 17q$  for some  $q \in \mathbf{Z}$ , and then  $13 + 17q \equiv 5 \pmod{13}$ . This reduces to  $4q \equiv 5 \pmod{13}$ , so  $40q \equiv 50 \pmod{13}$ , or  $q \equiv 11 \pmod{13}$ . This leads to the answer,  $x \equiv 13 + 17 \cdot 11 \equiv 200 \pmod{221}$ .

31. Solve the system of congruences  $x \equiv 5 \pmod{25}$   $x \equiv 23 \pmod{32}$ .

*Solution:* Write  $x = 23 + 32q$  for some  $q \in \mathbf{Z}$ , and substitute to get  $23 + 32q \equiv 5 \pmod{25}$ , which reduces to  $7q \equiv 7 \pmod{25}$ , so  $q \equiv 1 \pmod{25}$ . This gives  $x \equiv 55 \pmod{25 \cdot 32}$ .

32. Give integers  $a, b, m, n$  to provide an example of a system

$$x \equiv a \pmod{m} \quad x \equiv b \pmod{n}$$

that has no solution.

*Solution:* In the example the integers  $m$  and  $n$  cannot be relatively prime. This is the clue to take  $m = n = 2$ , with  $a = 1$  and  $b = 0$ .

33. (a) Compute the last digit in the decimal expansion of  $4^{100}$ .

*Solution:* The last digit is the remainder when divided by 10. Thus we must compute the congruence class of  $4^{100} \pmod{10}$ . We have  $4^2 \equiv 6 \pmod{10}$ , and then  $6^2 \equiv 6 \pmod{10}$ . Thus  $4^{100} = (4^2)^{50} \equiv 6^{50} \equiv 6 \pmod{10}$ .

(b) Is  $4^{100}$  divisible by 3?

*Solution:* No, since  $4^{100} \equiv 1^{100} \equiv 1 \pmod{3}$ . Or you can write  $2^{200}$  as the prime factorization, and then  $(3, 2^{200}) = 1$ .

34. Find all integers  $n$  for which  $13 \mid 4(n^2 + 1)$ .

*Solution:* This is equivalent solving the congruence  $4(n^2 + 1) \equiv 0 \pmod{13}$ . Since  $\gcd(4, 13) = 1$ , we can cancel 4, to get  $n^2 \equiv -1 \pmod{13}$ . Just computing the squares modulo 13 gives us  $(\pm 1)^2 = 1$ ,  $(\pm 2)^2 = 4$ ,  $(\pm 3)^2 = 9$ ,  $(\pm 4)^2 \equiv 3 \pmod{13}$ ,  $(\pm 5)^2 \equiv -1 \pmod{13}$ , and  $(\pm 6)^2 \equiv -3 \pmod{13}$ . We have done the computation for representatives of each congruence class, so the answer to the original question is  $x \equiv \pm 5 \pmod{13}$ .

35. Prove that  $10^{n+1} + 4 \cdot 10^n + 4$  is divisible by 9, for all positive integers  $n$ .

*Solution:* This could be proved by induction, but a more elegant proof can be given by simply observing that  $10^{n+1} + 4 \cdot 10^n + 4 \equiv 0 \pmod{9}$  since  $10 \equiv 1 \pmod{9}$ .

36. Prove that the fourth power of an integer can only have 0, 1, 5, or 6 as its units digit.

*Solution:* Since the question deals with the units digit of  $n^4$ , it is really asking to find  $n^4 \pmod{10}$ . All we need to do is to compute the fourth power of each congruence class modulo 10:  $0^4 = 0$ ,  $(\pm 1)^4 = 1$ ,  $(\pm 2)^4 = 16 \equiv 6 \pmod{10}$ ,  $(\pm 3)^4 = 81 \equiv 1 \pmod{10}$ ,  $(\pm 4)^4 \equiv 6^2 \equiv 6 \pmod{10}$ , and  $5^4 \equiv 5^2 \equiv 5 \pmod{10}$ . This shows that the only possible units digits for  $n^4$  are 0, 1, 5, and 6.