2

Functions

2.1 SOLUTIONS

20. The "Vertical Line Test" from calculus says that a curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once. Explain why this agrees with Definition 2.1.1.

    Solution: We assume that the x-axis is the domain and the y-axis is the codomain of the function that is to be defined by the given curve. According to Definition 2.1.1, a subset of the plane defines a function if for each element x in the domain there is a unique element y in the codomain such that (x, y) belongs to the subset of the plane. If a vertical line intersects the curve in two distinct points, then there will be points (x_1, y_1) and (x_2, y_2) on the curve with $x_1 = x_2$ and $y_1 \neq y_2$. Thus if we apply Definition 2.1.1 to the given curve, the uniqueness part of the definition translates directly into the "vertical line test".

21. The "Horizontal Line Test" from calculus says that a function is one-to-one if and only if no horizontal line intersects its graph more than once. Explain why this agrees with Definition 2.1.4.

Solution: If a horizontal line intersects the graph of the function more than once, then the points of intersection represent points (x_1, y_1) and (x_2, y_2) for which $x_1 \neq x_2$ but $y_1 = y_2$. According to Definition 2.1.4, a function is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Equivalently, if (x_1, y_1) and (x_2, y_2) line on its graph, then we cannot have $y_1 = y_2$ while $x_1 \neq x_2$. In this context, the "horizontal line test" is exactly the same as the condition given in Definition 2.1.4.

more than one

22. In calculus the graph of an inverse function f^{-1} is obtained by reflecting the graph of f about the line $y = x$. Explain why this agrees with Definition 2.1.7.

Solution: We first note that the reflection of a point (a, b) in the line $y = x$ is the point (b, a). This can be seen by observing that the line segment joining (a, b) and (b, a) has slope -1, which makes it perpendicular to the line $y = x$, and that this line segment intersects the line $y = x$ at the midpoint $((a + b)/2, (a + b)/2)$ of the segment.

If $f : \mathbf{R} \to \mathbf{R}$ has an inverse, and the point (x, y) lies on the graph of f, then $y = f(x)$, and so $f^{-1}(y) = f^{-1}(f(x)) = x$. This shows that the point (x, y) lies on the graph of f^{-1}. Conversely, if (x, y) lies on the graph of f^{-1}, then $x = f^{-1}(y)$, and therefore $y = f(f^{-1}(y)) = f(x)$, which shows that (y, x) lies on the graph of f.

On the other hand, suppose that the graph of the function g is defined by reflecting the graph of f in the line $y = x$. For any real number x, if $y = f(x)$ then we have $g(f(x)) = g(y) = x$ and for any real number y we have $f(g(y)) = f(x) = y$, where $x = g(y)$. This shows that $g = f^{-1}$, and so f has an inverse.

23. Let A be an $n \times n$ matrix with entries in $\mathbf{R}$. Define a linear transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ by $L(\mathbf{x}) = \mathbf{Ax}$, for all $\mathbf{x} \in \mathbf{R^n}$.

(a) Show that L is an invertible function if and only if $\det(A) \neq 0$.

Solution: I need to assume that you know that a square matrix A is invertible if and only if $\det(A) \neq 0$.

First, if L has an inverse, then it can also be described by multiplication by a matrix B, which must satisfy the conditions $BA = I$, and $AB = I$, where I is the $n \times n$ identity matrix. Thus A is an invertible matrix, and so $\det(A) \neq 0$.

On the other hand, if $\det(A) \neq 0$, then A is invertible, and so L has an inverse, defined by $L^{-1}(\mathbf{x}) = \mathbf{A^{-1}x}$, for all $\mathbf{x} \in \mathbf{R^n}$.

(b) Show that if L is either one-to-one or onto, then it is invertible.

Solution: The rank of the matrix A is the dimension of the column space of A, and this is the image of the transformation L, so L is onto if and only if A has rank n.

On the other hand, the nullity of A is the dimension of the solution space of the equation $A\mathbf{x} = \mathbf{0}$, and L is one-to-one if and only if the nullity of A is zero, since $A\mathbf{x_1} = \mathbf{Ax_2}$ if and only if $A(\mathbf{x_1} - \mathbf{x_2}) = \mathbf{0}$.

To prove part (b) we need to use the Rank–Nullity Theorem, which states that if A is an $n \times n$ matrix, then the rank of A plus the nullity of A is n. Since the matrix A is invertible if and only if it has rank n, it follows that L is invertible if and only if L is onto, and then the Rank–Nullity Theorem shows that this happens if and only if L is one-to-one.

24. Let A be an $m \times n$ matrix with entries in $\mathbf{R}$, and assume that $m > n$. Define a linear transformation $L : \mathbf{R}^n \to \mathbf{R}^m$ by $L(\mathbf{x}) = \mathbf{Ax}$, for all $\mathbf{x} \in \mathbf{R^n}$. Show that L is a one-to-one function if $\det(A^T A) \neq 0$, where A^T is the transpose of A.

    Solution: If $\det(A^T A) \neq 0$, then $A^T A$ is an invertible matrix. If we define $K : \mathbf{R}^m \to \mathbf{R}^n$ by $K(\mathbf{x}) = (\mathbf{A^T A})^{\mathbf{-1}}\mathbf{A^T x}$, for all $\mathbf{x} \in \mathbf{R^m}$, then KL is the identity function on $\mathbf{R}^m$. It then follows from Exercise 17 that L is one-to-one.

    Comment: There is a stronger result that depends on knowing a little more linear algebra. In some linear algebra courses it is proved that $\det(A^T A)$ gives the n-dimensional "content" of the parallepiped defined by the column vectors of A. This content is nonzero if and only if the vectors are linearly independent, and so $\det(A^T A) \neq 0$ if and only if the column vectors of A are linearly independent. According to the Rank–Nullity Theorem, this happens if and only if the nullity of A is zero. In other words, L is a one-to-one linear transformation if and only if $\det(A^T A) \neq 0$.

25. Let A be an $n \times n$ matrix with entries in $\mathbf{R}$. Define a linear transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ by $L(\mathbf{x}) = \mathbf{Ax}$, for all $\mathbf{x} \in \mathbf{R^n}$. Prove that L is one-to-one if and only if no eigenvalue of A is zero.

    Note: A vector $\mathbf{x}$ is called an eigenvector of A if it is nonzero and there exists a scalar λ such a that $A\mathbf{x} = \lambda\mathbf{x}$.

    Solution: As noted in the solution to problem 23, $A\mathbf{x_1} = \mathbf{Ax_2}$ if and only if $A(\mathbf{x_1} - \mathbf{x_2}) = \mathbf{0}$, and so L is one-to-one if and only if $A\mathbf{x} \neq \mathbf{0}$ for all nonzero vectors $\mathbf{x}$. This is equivalent to the statement that there is no nonzero vector $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0} \cdot \mathbf{x}$, which translates into the given statement about eigenvalues of A.

26. Let a be a fixed element of $\mathbf{Z}_{17}^\times$. Define the function $\theta : \mathbf{Z}_{17}^\times \to \mathbf{Z}_{17}^\times$ by $\theta(x) = ax$, for all $x \in \mathbf{Z}_{17}^\times$. Is θ one to one? Is θ onto? If possible, find the inverse function θ^{-1}.

    Solution: Since a has an inverse in $\mathbf{Z}_{17}^\times$, we can define $\psi : \mathbf{Z}_{17}^\times \to \mathbf{Z}_{17}^\times$ by $\psi(x) = a^{-1}x$, for all $x \in \mathbf{Z}_{17}^\times$. Then $\psi(\theta(x)) = \psi(ax) = a^{-1}(ax) = (a^{-1}a)x = x$ and $\theta(\psi(x)) = \theta(a^{-1}x) = a(a^{-1}x) = (aa^{-1})x = x$, which shows that $\psi = \theta^{-1}$. This implies that θ is one-to-one and onto.