# 2.2 SOLUTIONS

14. On the set $\{(a,b)\}$ of all ordered pairs of positive integers, define $(x_1,y_1) \sim (x_2,y_2)$ if $x_1y_2 = x_2y_1$. Show that this defines an equivalence relation.

    *Solution:* We first show that the reflexive law holds. Given an ordered pair $(a,b)$, we have $ab = ba$, and so $(a,b) \sim (a,b)$.

    We next check the symmetric law. Given $(a_1,b_1)$ and $(a_2,b_2)$ with $(a_1,b_1) \sim (a_2,b_2)$, we have $a_1b_2 = a_2b_1$, and so $a_2b_1 = a_1b_2$, which shows that $(a_2,b_2) \sim (a_1,b_1)$.

    Finally, we verify the transitive law. Given $(a_1,b_1)$, $(a_2,b_2)$, and $(a_3,b_3)$ with $(a_1,b_1) \sim (a_2,b_2)$ and $(a_2,b_2) \sim (a_3,b_3)$, we have the equations $a_1b_2 = a_2b_1$ and $a_2b_3 = a_3b_2$. If we multiply the first equation by $b_3$ and the second equation by $b_1$, we get $a_1b_2b_3 = a_2b_1b_3 = a_3b_1b_2$. Since $b_2 \neq 0$ we can cancel to obtain $a_1b_3 = a_3b_1$, showing that $(a_1,b_1) \sim (a_3,b_3)$.

15. On the set $\mathbf{C}$ of complex numbers, define $z_1 \sim z_2$ if $||z_1|| = ||z_2||$. Show that $\sim$ is an equivalence relation.

    *Solution:* The reflexive, symmetric, and transitive laws can be easily verified since $\sim$ is defined in terms of an equality, and equality is itself an equivalence relation.

16. Let $\mathbf{u}$ be a fixed vector in $\mathbf{R}^3$, and assume that $\mathbf{u}$ has length 1. For vectors $\mathbf{v}$ and $\mathbf{w}$, define $\mathbf{v} \sim \mathbf{w}$ if $\mathbf{v}\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{u}$, where $\cdot$ denotes the standard dot product. Show that $\sim$ is an equivalence relation, and give a geometric description of the equivalence classes of $\sim$.

    *Solution:* The reflexive, symmetric, and transitive laws for the relation $\sim$ really depend on an equality, and can easily be verified. Since $\mathbf{u}$ has length 1, $\mathbf{v}\cdot\mathbf{u}$ represents the length of the projection of $\mathbf{v}$ onto the line determined by $\mathbf{u}$. Thus two vectors are equivalent if and only if they lie in the same plane perpendicular to $\mathbf{u}$. It follows that the equivalence classes of $\sim$ are the planes in $\mathbf{R}^3$ that are perpendicular to $\mathbf{u}$.

17. For the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$, for all $x \in \mathbf{R}$, describe the equivalence relation on $\mathbf{R}$ that is determined by $f$.

    *Solution:* The equivalence relation determined by $f$ is defined by setting $a \sim b$ if $f(a) = f(b)$, so $a \sim b$ if and only if $a^2 = b^2$, or, $a \sim b$ if and only if $|a| = |b|$.

18. For the linear transformation $L : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

    $$L(x,y,z) = (x+y+z, x+y+z, x+y+z) ,$$

    for all $(x,y,z) \in \mathbf{R}^3$, give a geometric description of the partition of $\mathbf{R}^3$ that is determined by $L$.

*Solution:* Since $(a_1, a_2, a_3) \sim (b_1, b_2, b_3)$ if $L(a_1, a_2, a_3) = L(b_1, b_2, b_3)$, it follows from the definition of $L$ that $(a_1, a_2, a_3) \sim (b_1, b_2, b_3)$ if and only if $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$. For example, $\{(x, y, z) \mid L(x, y, z) = (0, 0, 0)\}$ is the plane through the origin whose equation is $x + y + z = 0$, with normal vector $(1, 1, 1)$. The other subsets in the partition of $\mathbf{R}^3$ defined by $L$ are planes parallel to this one. Thus the partition consists of the planes perpendicular to the vector $(1, 1, 1)$.

19. Define the formula $f : \mathbf{Z}_{12} \to \mathbf{Z}_{12}$ by $f([x]_{12}) = [x]_{12}^2$, for all $[x]_{12} \in \mathbf{Z}_{12}$. Show that the formula $f$ defines a function. Find the image of $f$ and the set $\mathbf{Z}_{12}/f$ of equivalence classes determined by $f$.

   *Solution:* The formula for $f$ is well-defined since if $[x_1]_{12} = [x_2]_{12}$, then $x_1 \equiv x_2 \pmod{12}$, and so $x_1^2 \equiv x_2^2 \pmod{12}$, which shows that $f([x_1]_{12}) = f([x_2]_{12})$.

   To compute the images of $f$ we have $[0]_{12}^2 = [0]_{12}$, $[\pm 1]_{12}^2 = [1]_{12}$, $[\pm 2]_{12}^2 = [4]_{12}$, $[\pm 3]_{12}^2 = [9]_{12}$, $[\pm 4]_{12}^2 = [4]_{12}$, $[\pm 5]_{12}^2 = [1]_{12}$, and $[6]_{12}^2 = [0]_{12}$. Thus $f(\mathbf{Z}_{12}) = \{[0]_{12}, [1]_{12}, [4]_{12}, [9]_{12}\}$. The corresponding equivalence classes determined by $f$ are $\{[0]_{12}, [6]_{12}\}$, $\{[\pm 1]_{12}, [\pm 5]_{12}\}$, $\{[\pm 2]_{12}, [\pm 4]_{12}\}$, $\{[\pm 3]_{12}\}$.

20. On the set of all $n \times n$ matrices over $\mathbf{R}$, define $A \sim B$ if there exists an invertible matrix $P$ such that $PAP^{-1} = B$. Check that $\sim$ defines an equivalence relation.

   *Solution:* We have $A \sim A$ since $IAI^{-1} = A$, where $I$ is the $n \times n$ identity matrix. If $A \sim B$, then $PAP^{-1} = B$ for some invertible matrix $P$, and so we get $A = P^{-1}B(P^{-1})^{-1}$. If $A \sim B$ and $B \sim C$, then $PAP^{-1} = B$ and $QBQ^{-1} = C$ for some $P, Q$. Substituting gives $Q(PAP^{-1})Q^{-1} = (QP)A(QP)^{-1} = C$, and so $A \sim C$.