2.3 SOLUTIONS

13. For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 5 & 6 & 9 & 2 & 4 & 8 & 1 & 3 \end{pmatrix}$, write σ as a product of disjoint cycles. What is the order of σ? Is σ an even permutation? Compute σ^{-1}.

    Solution: We have $\sigma = (1,7,8)(2,5)(3,6,4,9)$, and so its order is 12 since $\text{lcm}[3,2,4] = 12$. It is an even permutation, since it can be expressed as the product of 6 transpositions. We have $\sigma^{-1} = (1,8,7)(2,5)(3,9,4,6)$.

14. For the permutations $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 1 & 8 & 3 & 6 & 4 & 7 & 9 \end{pmatrix}$ and

    $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 5 & 4 & 7 & 2 & 6 & 8 & 9 & 3 \end{pmatrix}$, write each of these permutations as a product of disjoint cycles: σ, τ, $\sigma\tau$, $\sigma\tau\sigma^{-1}$, σ^{-1}, τ^{-1}, $\tau\sigma$, $\tau\sigma\tau^{-1}$.

    Solution: $\sigma = (1,2,5,3)(4,8,7)$; $\tau = (2,5)(3,4,7,8,9)$; $\sigma\tau = (1,2,3,8,9)$; $\sigma\tau\sigma^{-1} = (1,8,4,7,9)(3,5)$; $\sigma^{-1} = (1,3,5,2)(4,7,8)$; $\tau^{-1} = (2,5)(3,9,8,7,4)$; $\tau\sigma = (1,5,4,9,3)$; $\tau\sigma\tau^{-1} = (1,5,2,4)(7,9,8)$.

15. Let $\sigma = (2,4,9,7,)(6,4,2,5,9)(1,6)(3,8,6) \in S_9$. Write σ as a product of disjoint cycles. What is the order of σ? Compute σ^{-1}.

    Solution: We have $\sigma = (1,9,6,3,8)(2,5,7)$, so it has order $15 = \text{lcm}[5,3]$, and $\sigma^{-1} = (1,8,3,6,9)(2,7,5)$.

16. Compute the order of $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 2 & 11 & 4 & 6 & 8 & 9 & 10 & 1 & 3 & 5 \end{pmatrix}$. For $\sigma = (3,8,7)$, compute the order of $\sigma\tau\sigma^{-1}$.

    Solution: Since $\tau = (1,7,9)(3,11,5,6,8,10)$, it has order 6. We have $\sigma\tau\sigma^{-1} = (3,8,7)(1,7,9)(3,11,5,6,8,10)(3,7,8) = (1,3,9)(8,11,5,6,7,10)$, so the cycle structure of $\sigma\tau\sigma^{-1}$ is the same as that of τ, and thus $\sigma\tau\sigma^{-1}$ has order 6.

17. Prove that if $\tau \in \mathcal{S}_n$ is a permutation with order m, then $\sigma\tau\sigma^{-1}$ has order m, for any permutation $\sigma \in \mathcal{S}_n$.

    Solution: Assume that $\tau \in \mathcal{S}_n$ has order m. It follows from the identity $(\sigma\tau\sigma^{-1})^k = \sigma\tau^k\sigma^{-1}$ that $(\sigma\tau\sigma^{-1})^m = \sigma\tau^m\sigma^{-1} = \sigma(1)\sigma^{-1} = (1)$. On the other hand, the order of $\sigma\tau\sigma^{-1}$ cannot be less than n, since $(\sigma\tau\sigma^{-1})^k = (1)$ implies $\sigma\tau^k\sigma^{-1} = (1)$, and then $\tau^k = \sigma^{-1}\sigma = (1)$.

18. Show that S_{10} has elements of order 10, 12, and 14, but not 11 or 13.

    Solution: The permutation $(1,2)(3,4,5,6,7)$ has order 10, while the element $(1,2,3)(4,5,6,7)$ has order 12, and $(1,2)(3,4,5,6,7,8,9)$ has order 14. On the other hand, since 11 and 13 are prime, any element of order 11 or 13 would have to be a cycle, and there are no cycles of that length in S_{10}.

19. Let S be a set, and let X be a subset of S. Let $G = \{\sigma \in \text{Sym}(S) \mid \sigma(X) \subset X\}$. Prove that G is a group of permutations.

20. Let G be a group of permutations, with $G \subseteq \text{Sym}(S)$, for the set S. Let τ be a fixed permutation in $\text{Sym}(S)$. Prove that

$$\tau G \tau^{-1} = \{\sigma \in \text{Sym}(S) \mid \sigma = \tau\gamma\tau \text{ for some } \gamma \in G\}$$

is a group of permutations.