SOLUTIONS TO THE REVIEW PROBLEMS

1. For the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$, for all $x \in \mathbf{R}$, describe the equivalence relation on $\mathbf{R}$ that is determined by f.

2. Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^3 + 3xz - 5$, for all $x \in \mathbf{R}$. Show that f is a one-to-one function.

   Hint: Use the derivative of f to show that f is a strictly increasing function.

3. On the set $\mathbf{Q}$ of rational numbers, define $x \sim y$ if $x - y$ is an integer. Show that $\sim$ is an equivalence relation.

4. In S_{10}, let $\alpha = (1,3,5,7,9)$, $\beta = (1,2,6)$, and $\gamma = (1,2,5,3)$. For $\sigma = \alpha\beta\gamma$, write σ as a product of disjoint cycles, and use this to find its order and its inverse. Is σ even or odd?

   Solution: We have $\sigma = (1,6,3,2,7,9)$, so σ has order 6, and

   $\sigma^{-1} = (1,9,7,2,3,6)$. Since σ has length 6, it can be written as a product of 5 transpositions, so it is an odd permutation.

5. Define the function $\phi : \mathbf{Z}_{17}^\times \to \mathbf{Z}_{17}^\times$ by $\phi(x) = x^{-1}$, for all $x \in \mathbf{Z}_{17}^\times$. Is ϕ one to one? Is ϕ onto? If possible, find the inverse function ϕ^{-1}.

   Solution: For all $x \in \mathbf{Z}_{17}^\times$ we have $\phi(\phi(x)) = \phi(x^{-1}) = (x^{-1})^{-1} = x$, so $\phi = \phi^{-1}$, which also shows that ϕ is one-to-one and onto.

6. (a) Let α be a fixed element of S_n. Show that $\phi_\alpha : S_n \to S_n$ defined by $\phi_\alpha(\sigma) = \alpha\sigma\alpha^{-1}$, for all $\sigma \in S_n$, is a one-to-one and onto function.

   Solution: If $\phi_\alpha(\sigma) = \phi_\alpha(\tau)$, for $\sigma, \tau \in S_n$, then $\alpha\sigma\alpha^{-1} = \alpha\tau\alpha^{-1}$. We can multiply on the left by α^{-1} and on the right by α, to get $\sigma = \tau$, so ϕ_α is one-to-one. Finally, given $\tau \in S_n$, we have $\phi_\alpha(\sigma) = \tau$ for $\sigma = \alpha^{-1}\tau\alpha$, and so ϕ_α is onto.

   Another way to show that ϕ_α is one-to-one and onto is to show that it has an inverse function. A short computation shows that $(\phi_\alpha)^{-1} = \phi_{\alpha^{-1}}$.

   (b) In S_3, let $\alpha = (1,2)$. Compute ϕ_α.

   Solution: Since $(1,2)$ is its own inverse, direct computations show that

   $\phi_\alpha((1)) = (1)$, $\phi_\alpha((1,2)) = (1,2)$, $\phi_\alpha((1,3)) = (2,3)$, $\phi_\alpha((2,3)) = (1,3)$,

   $\phi_\alpha((1,2,3)) = (1,3,2)$, and $\phi_\alpha((1,3,2)) = (1,2,3)$.