3.6 SOLUTIONS

22. In the dihedral group $\mathcal{D}_n = \{a^i b^j \mid 0 \le i < n,\ 0 \le j < 2\}$ with $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$, show that $ba^i = a^{n-i}b$, for all $0 \le i < n$.

    Solution: For $i = 1$, the equation $ba^i = a^{n-i}b$ is just the relation that defines the group. If we assume that the result holds for $i = k$, then for $i = k + 1$ we have

    $$ba^{k+1} = (ba^k)a = (a^{n-k}b)a = a^{n-k}(ba) = a^{n-k}a^{-1}b = a^{n-(k+1)}b\ .$$

    This implies that the result must hold for all i with $0 \le i < n$.

    Comment: This is similar to a proof by induction, but for each given n we only need to worry about a finite number of equations.

23. In the dihedral group $\mathcal{D}_n = \{a^i b^j \mid 0 \le i < n,\ 0 \le j < 2\}$ with $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$, show that each element of the form $a^i b$ has order 2.

    Solution: Using the result from the previous problem, we have $(a^i b)^2 = (a^i b)(a^i b) = a^i(ba^i)b = a^i(a^{n-i}b)b = (a^i a^{n-i})(b^2) = a^n e = e$.

24. In $\mathcal{S}_4$, find the subgroup H generated by $(1, 2, 3)$ and $(1, 2)$.

    Solution: Let $a = (1, 2, 3)$ and $b = (1, 2)$. Then H must contain $a^2 = (1, 3, 2)$, $ab = (1, 3)$ and $a^2b = (2, 3)$, and this set of elements is closed under multiplication. (We have just listed the elements of $\mathcal{S}_3$.) Thus $H = \{(1), a, a^2, b, ab, a^2b\} = \{(1),\ (1, 2, 3),\ (1, 3, 2),\ (1, 2),\ (1, 3),\ (2, 3)\}$.

25. For the subgroup H of $\mathcal{S}_4$ defined in the previous problem, find the corresponding subgroup $\sigma H \sigma^{-1}$, for $\sigma = (1, 4)$.

    Solution: We need to compute $\sigma\tau\sigma^{-1}$, for each $\tau \in H$. Since $(1, 4)^{-1} = (1, 4)$, we have $(1, 4)(1)(1, 4) = (1)$, and $(1, 4)(1, 2, 3)(1, 4) = (2, 3, 4)$. As a shortcut, we can use Exercise 2.3.10, which shows that $\sigma(1, 2, 3)\sigma^{-1} = (\sigma(1), \sigma(2), \sigma(3)) = (4, 2, 3)$. Then we can quickly do the other computations:

    $$\begin{aligned}(1,4)(1,3,2)(1,4)^{-1} &= (4,3,2)\\ (1,4)(1,2)(1,4)^{-1} &= (4,2)\\ (1,4)(1,3)(1,4)^{-1} &= (4,3)\\ (1,4)(2,3)(1,4)^{-1} &= (2,3).\end{aligned}$$

    Thus $(1, 4)H(1, 4)^{-1} = \{(1),\ (2, 3, 4),\ (2, 4, 3),\ (2, 3),\ (2, 4),\ (3, 4)\}$.

26. Show that each element in $\mathcal{A}_4$ can be written as a product of 3-cycles.

    Solution: We first list the 3-cycles: $(1, 2, 3)$, $(1, 2, 4)$, $(1, 3, 2)$, $(1, 3, 4)$, $(1, 4, 2)$, $(1, 4, 3)$, $(2, 3, 4)$, and $(2, 4, 3)$. Rather than starting with each of the other elements and then trying to write them as a product of 3-cycles, it is easier to just look at the possible products of 3-cycles. We have $(1,2,3)(1,2,4) = (1,3)(2,4)$, $(1,2,4)(1,2,3) = (1,4)(2,3)$, $(1,2,3)(2,3,4) = (1,2)(3,4)$, and this accounts for all 12 of the elements in $\mathcal{A}_4$.

27. In the dihedral group $\mathcal{D}_n = \{a^i b^j \mid 0 \leq i < n,\ 0 \leq j < 2\}$ with $o(a) = n$, $o(b) = 2$, and $ba = a^{-1}b$, find the centralizer of a.

    Solution: The centralizer $C(a)$ contains all powers of a, so we have $\langle a \rangle \subseteq C(a)$. This shows that $C(a)$ has at least n elements. On the other hand, $C(a) \neq \mathcal{D}_n$, since by definition b does not belong to $C(a)$. Since $\langle a \rangle$ contains exactly half of the elements in $\mathcal{D}_n$, Lagrange's theorem show that there is no subgroup that lies strictly between $\langle a \rangle$ and $\mathcal{D}_n$, so $\langle a \rangle \subseteq C(a) \subseteq \mathcal{D}_n$ and $C(a) \neq \mathcal{D}_n$ together imply that $C(a) = \langle a \rangle$.

28. Find the centralizer of $(1,2,3)$ in $\mathcal{S}_3$, in $\mathcal{S}_4$, and in $\mathcal{A}_4$.

    Solution: Since any power of an element a commutes with a, the centralizer $C(a)$ always contains the cyclic subgroup $\langle a \rangle$ generated by a. Thus the centralizer of $(1,2,3)$ always contains the subgroup $\{(1),(1,2,3),(1,3,2)\}$.

    In $\mathcal{S}_3$, the centralizer of $(1,2,3)$ is equal to $\langle\, (1,2,3)\, \rangle$, since it is easy to check that $(1,2)$ does not belong to the centralizer, and by Lagrange's theorem a proper subgroup of a group with 6 elements can have at most 3 elements. To find the centralizer of $(1,2,3)$ in $\mathcal{S}_4$ we have to work a bit harder.

    It helps to have some shortcuts when doing the necessary computations. To see that x belongs to $C(a)$, we need to check that $xa = ax$, or that $axa^{-1} = x$. Exercise 2.3.10 provides a quick way to do this in a group of permutations. That exercise shows that if $(1,2,\ldots,k)$ is a cycle of length k and σ is any permutation, then $\sigma(1,2,\ldots,k)\sigma^{-1} = (\sigma(1),\sigma(2),\ldots,\sigma(k))$.

    Let $a = (1,2,3)$. From the computations in $\mathcal{S}_3$, we know that $(1,2)$, $(1,3)$, and $(2,3)$ do not commute with a. The remaining transpositions in $\mathcal{S}_4$ are $(1,4)$, $(2,4)$, and $(3,4)$. Using Exercise 2.3.10, we have $a(1,4)a^{-1} = (2,4)$, $a(2,4)a^{-1} = (3,4)$, and $a(3,4)a^{-1} = (1,4)$, so no transposition in $\mathcal{S}_4$ commutes with a. For the products of the transposition, we have $a(1,2)(3,4)a^{-1} = (2,3)(1,4)$, $a(1,3)(2,4)a^{-1} = (2,1)(3,4)$, and $a(1,4)(2,3)a^{-1} = (2,4)(3,1)$, and so no product of transpositions belongs to $C(a)$.

    If we do a similar computation with a 4-cycle, we will have $a(x,y,z,4)a^{-1} = (u,v,w,4)$, since a just permutes the numbers x, y, and z. This means that $w \neq z$, so $(u,v,w,4) \neq (x,y,z,4)$. Without doing all of the calculations, we can conclude that no 4-cycle belongs to $C(a)$. This accounts for an additional 6 elements. A similar argument shows that no 3-cycle that includes the number 4 as one of its entries can belong to $C(a)$. Since there are 6 elements of this form, we now have a total of 21 elements that are not in $C(a)$, and therefore $C(a) = \langle a \rangle$. Finally, in $\mathcal{A}_4$ we must get the same answer: $C(a) = \langle a \rangle$.