

3.8 SOLUTIONS

27. List the cosets of $\langle 7 \rangle$ in \mathbf{Z}_{16}^\times . Is the factor group $\mathbf{Z}_{16}^\times / \langle 7 \rangle$ cyclic?

Solution: $\mathbf{Z}_{16}^\times = \{1, 3, 5, 7, 9, 11, 13, 15\}$.

$$\langle 7 \rangle = \{1, 7\} \quad 3 \langle 7 \rangle = \{3, 5\} \quad 9 \langle 7 \rangle = \{9, 15\} \quad 11 \langle 7 \rangle = \{11, 13\}$$

Since $3^2 \notin \langle 7 \rangle$, the coset $3 \langle 7 \rangle$ does not have order 2, so it must have order 4, showing that the factor group is cyclic.

28. Let $G = \mathbf{Z}_6 \times \mathbf{Z}_4$, let $H = \{(0, 0), (0, 2)\}$, and let $K = \{(0, 0), (3, 0)\}$.

(a) List all cosets of H ; list all cosets of K .

Solution: The cosets of $H = \{(0, 0), (0, 2)\}$ are

$$(0, 0) + H = \{(0, 0), (0, 2)\} \quad (1, 0) + H = \{(1, 0), (1, 2)\}$$

$$(2, 0) + H = \{(2, 0), (2, 2)\} \quad (3, 0) + H = \{(3, 0), (3, 2)\}$$

$$(4, 0) + H = \{(4, 0), (4, 2)\} \quad (5, 0) + H = \{(5, 0), (5, 2)\}$$

$$(0, 1) + H = \{(0, 1), (0, 3)\} \quad (1, 1) + H = \{(1, 1), (1, 3)\}$$

$$(2, 1) + H = \{(2, 1), (2, 3)\} \quad (3, 1) + H = \{(3, 1), (3, 3)\}$$

$$(4, 1) + H = \{(4, 1), (4, 3)\} \quad (5, 1) + H = \{(5, 1), (5, 3)\}$$

The cosets of $K = \{(0, 0), (3, 0)\}$ are

$$(0, 0) + K = \{(0, 0), (3, 0)\} \quad (0, 1) + K = \{(0, 1), (3, 1)\}$$

$$(0, 2) + K = \{(0, 2), (3, 2)\} \quad (0, 3) + K = \{(0, 3), (3, 3)\}$$

$$(1, 0) + K = \{(1, 0), (4, 0)\} \quad (1, 1) + K = \{(1, 1), (4, 1)\}$$

$$(1, 2) + K = \{(1, 2), (4, 2)\} \quad (1, 3) + K = \{(1, 3), (4, 3)\}$$

$$(2, 0) + K = \{(2, 0), (5, 0)\} \quad (2, 1) + K = \{(2, 1), (5, 1)\}$$

$$(2, 2) + K = \{(2, 2), (5, 2)\} \quad (2, 3) + K = \{(2, 3), (5, 3)\}$$

(b) You may assume that any abelian group of order 12 is isomorphic to either \mathbf{Z}_{12} or $\mathbf{Z}_6 \times \mathbf{Z}_2$. Which answer is correct for G/H ? For G/K ?

Solution: Adding an element of G to itself 6 times yields a 0 in the first component and either 0 or 2 in the second component, producing an element in H . Thus the order of an element in G/H is at most 6, and so $G/H \cong \mathbf{Z}_6 \times \mathbf{Z}_2$.

On the other hand, $(1, 1) + K$ has order 12 in G/K , and so $G/K \cong \mathbf{Z}_{12}$.

29. Let the dihedral group D_n be given via generators and relations, with generators a of order n and b of order 2, satisfying $ba = a^{-1}b$.

(a) Show that $ba^i = a^{-i}b$ for all i with $1 \leq i < n$.

Solution: The identity holds for all positive integers i , and can be proved inductively: assuming $ba^k = a^{-k}b$, we have $ba^{k+1} = ba^k a = a^{-k}ba = a^{-k}a^{-1}b = a^{-(k+1)}b$.

(b) Show that any element of the form $a^i b$ has order 2.

Solution: We have $(a^i b)^2 = a^i b a^i b = a^i a^{-i} b^2 = a^0 = e$.

(c) List all left cosets and all right cosets of $\langle b \rangle$

Solution: The left cosets of $\langle b \rangle$ have the form $a^i \langle b \rangle = \{a^i, a^i b\}$, for $0 \leq i < n$.

The right cosets of $\langle b \rangle$ have the form $\langle b \rangle a^i = \{a^i, a^{-i} b\}$, for $0 \leq i < n$.

30. Let $G = D_6$ and let N be the subgroup $\langle a^3 \rangle = \{e, a^3\}$ of G .

(a) Show that N is a normal subgroup of G .

Solution: The argument is the same as in the previous problem.

(b) Is G/N abelian?

Solution: For $aN = \{a, a^4\}$ and $bN = \{b, a^3 b\}$, we have $(aN)(bN) = abN = \{ab, a^4 b\}$, while $(bN)(aN) = baN = a^5 bN = \{a^5 b, a^2 b\}$. Thus $(aN)(bN) \neq (bN)(aN)$, and G/N is not abelian.

31. Let G be the dihedral group D_{12} , and let $N = \{e, a^3, a^6, a^9\}$.

(a) Prove that N is a normal subgroup of G , and list all cosets of N .

Solution: Since $N = \langle a^3 \rangle$, it is a subgroup. It is normal since $a^i (a^{3n}) a^{-i} = a^{3n}$ and $a^i b (a^{3n}) a^i b = a^i a^{-3n} a^{-i} = (a^{3n})^{-1}$. (We are using the fact that $ba^i = a^{-i} b$.)

The cosets of N are

$$\begin{aligned} N &= \{e, a^3, a^6, a^9\}, & Nb &= \{ab, a^3 b, a^6 b, a^9 b\}, \\ Na &= \{a, a^4, a^7, a^{10}\}, & Nab &= \{ab, a^4 b, a^7 b, a^{10} b\}, \\ Na^2 &= \{a^2, a^5, a^8, a^{11}\}, & Na^2 b &= \{a^2 b, a^5 b, a^8 b, a^{11} b\}. \end{aligned}$$

(b) You may assume that G/N is isomorphic to either \mathbf{Z}_6 or S_3 . Which is correct?

Solution: The factor group G/N is not abelian, since $NaNb = Nab$ but $NbNa = Na^2b$, because $ba = a^{11} b \in Na^2b$. Thus $G/N \cong S_3$.

32. (a) Let G be a group. For $a, b \in G$ we say that b is conjugate to a , written $b \sim a$, if there exists $g \in G$ such that $b = gag^{-1}$. Show that \sim is an equivalence relation on G . The equivalence classes of \sim are called the *conjugacy classes* of G .

Solution: We have $a \sim a$ since we can use $g = e$. If $b \sim a$, the $b = gag^{-1}$ for some $g \in G$, and so $a = g^{-1}bg = g^{-1}b(g^{-1})^{-1}$, which shows that $a \sim b$. If $c \sim b$ and $b \sim a$, then $c = gbg^{-1}$ and $b = hah^{-1}$ for some $g, h \in G$, so $c = g(hah^{-1})g^{-1} = (gh)a(gh)^{-1}$, which shows that $c \sim a$. Thus \sim is an equivalence relation.

(b) Show that a subgroup N of G is normal in G if and only if N is a union of conjugacy classes.

Solution: The subgroup N is normal in G if and only if $a \in N$ implies $gag^{-1} \in G$, for all $g \in G$. Thus N is normal if and only if whenever it contains an element a it also contains the conjugacy class of a . Another way to say this is that N is a union of conjugacy classes.

33. Find the conjugacy classes of D_4 .

Solution: Remember: the notion of a conjugacy class was just defined in the previous exercise. Let $D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, with $a^4 = e$, $b^2 = e$, and $ba = a^{-1}b$. Since $xex^{-1} = e$, the only element conjugate to e is e itself.

If x is any power of a , then x commutes with a , and so $xax^{-1} = a$. If $x = a^i b$, then $xax^{-1} = a^i b a a^{-i} b = a^i a^{i-1} b^2 = a^{2i-1}$, so this shows that a^3 is the only conjugate of a (other than a itself).

The solution of an earlier problem shows that $xa^2x^{-1} = a^2$ in D_4 , so a^2 is not conjugate to any other element.

If $x = a^i$, then $xbx^{-1} = a^i b a^{-i} = a^i a^i b = a^{2i} b$. If $x = a^i b$, then $xbx^{-1} = (a^i b) b (a^i b)^{-1} = a^i a^i b = a^{2i} b$. Thus $a^2 b$ is the only conjugate of b .

If $x = a^i$, then $x(ab)x^{-1} = a^i a b a^{-i} = a^{i+1} a^i b = a^{2i+1} b$. If $x = a^i b$, then $xabx^{-1} = (a^i b) a b (a^i b)^{-1} = a^i a^{-1} a^i b = a^{2i-1} b$. Thus $a^3 b$ is the only conjugate of ab .

34. Let G be a group, and let N and H be subgroups of G such that N is normal in G .

(a) Prove that HN is a subgroup of G .

Solution: See Proposition 3.3.2. It is clear that $e = e \cdot e$ belongs to the set HN , so HN is nonempty. Suppose that x, y belong to HN . Then $x = h_1 n_1$ and $y = h_2 n_2$, for some $h_1, h_2 \in H$ and some $n_1, n_2 \in N$. We have

$$xy^{-1} = h_1 n_1 (h_2 n_2)^{-1} = h_1 n_1 n_2^{-1} h_2^{-1} = (h_1 h_2^{-1})(h_2 (n_1 n_2^{-1}) h_2^{-1}),$$

and this element belongs to HN since the assumption that N is normal guarantees that $h_2 (n_1 n_2^{-1}) h_2^{-1} \in N$.

(b) Prove that N is a normal subgroup of HN .

Solution: Since N is normal in G , it is normal in the subgroup HN , which contains it.

(c) Prove that if $H \cap N = \{e\}$, then HN/N is isomorphic to H .

Solution: Define $\phi : H \rightarrow HN/N$ by $\phi(x) = xN$ for all $x \in H$. (Defining a function from HN/N into H is more complicated.) Then $\phi(xy) = xyN = xNyN = \phi(x)\phi(y)$ for all $x, y \in H$. Any coset of N in HN has the form hnN for some $h \in H$ and some $n \in N$. But then $hnN = hN = \phi(h)$, and so this shows that ϕ is onto. Finally, ϕ is one-to-one since if $h \in H$ belongs to the kernel of ϕ , then $hN = \phi(h) = N$, and so $h \in N$. By assumption, $H \cap N = \{e\}$, and so $h = e$.