# 6

# Fields

## SOLUTIONS TO THE REVIEW PROBLEMS

1. Let $u$ be a root of the polynomial $x^3+3x+3$. In $\mathbf{Q}(u)$, express $(7-2u+u^2)^{-1}$ in the form $a+bu+cu^2$.

   *Solution:* Dividing $x^3+3x+3$ by $x^2-2x+7$ gives the quotient $x+2$ and remainder $-11$. Thus $u^3+3u+3=(u+2)(u^2-2u+7)-11$, and so $(7-2u+u^2)^{-1}=(2+u)/11=(2/11)+(1/11)u$.

2. (a) Show that $\mathbf{Q}(\sqrt{2}+i)=\mathbf{Q}(\sqrt{2},i)$.

   *Solution:* Let $u=\sqrt{2}+i$. Since $(\sqrt{2}+i)(\sqrt{2}-i)=2-i^2=3$, we have $\sqrt{2}-i=3(\sqrt{2}+i)^{-1}\in\mathbf{Q}(u)$, and it follows easily that $\sqrt{2}\in\mathbf{Q}(u)$ and $i\in\mathbf{Q}(u)$, so $\mathbf{Q}(\sqrt{2},i)\subseteq\mathbf{Q}(u)$. The reverse inclusion is obvious.

   (b) Find the minimal polynomial of $\sqrt{2}+i$ over $\mathbf{Q}$.

   *Solution:* We have $\mathbf{Q}\subseteq\mathbf{Q}(\sqrt{2})\subseteq\mathbf{Q}(\sqrt{2},i)$. Thus $[\mathbf{Q}(\sqrt{2}):\mathbf{Q}]=2$ since $\sqrt{2}$ is a root of a polynomial of degree 2 but is not in $\mathbf{Q}$. We have $[\mathbf{Q}(\sqrt{2},i):\mathbf{Q}(\sqrt{2})]=2$ since $i$ is a root of a polynomial of degree 2 over $\mathbf{Q}(\sqrt{2})$ but is not in $\mathbf{Q}(\sqrt{2})$. Thus $[\mathbf{Q}(\sqrt{2}+i):\mathbf{Q}]=4$, and so the minimal polynomial for $\sqrt{2}+i$ must have degree 4.

Since $u = \sqrt{2} + i$, we have $u - i = \sqrt{2}$, $u^2 - 2iu + i^2 = 2$, and $u^2 - 3 = 2iu$. Squaring again and combining terms gives $u^4 - 2u^2 + 9 = 0$. Thus the minimal polynomial for $\sqrt{2} + i$ is $x^4 - 2x^2 + 9$.

3. Find the minimal polynomial of $1 + \sqrt[3]{2}$ over $\mathbf{Q}$.

   *Solution:* Let $x = 1 + \sqrt[3]{2}$. Then $x - 1 = \sqrt[3]{2}$, and so $(x-1)^3 = 2$, which yields $x^3 - 3x^2 + 3x - 1 = 2$, and therefore $x^3 - 3x^2 + 3x - 3 = 0$. Eisenstein's criterion (with $p = 3$) shows that $x^3 - 3x^2 + 3x - 3$ is irreducible over $\mathbf{Q}$, so this is the required minimal polynomial.

4. Show that $x^3 + 6x^2 - 12x + 2$ is irreducible over $\mathbf{Q}$, and remains irreducible over $\mathbf{Q}(\sqrt[5]{2})$.

   *Solution:* Eisenstein's criterion works with $p = 2$. Since $x^5 - 2$ is also irreducible by Eisenstein's criterion, $[\mathbf{Q}(\sqrt[5]{2}) : \mathbf{Q}] = 5$. If $x^3 + 6x^2 - 12x + 2$ could be factored over $\mathbf{Q}(\sqrt[5]{2})$, then it would have a linear factor, and so it would have a root in $\mathbf{Q}(\sqrt[5]{2})$. This root would have degree 3 over $\mathbf{Q}$, and that is impossible since 3 is not a divisor of 5.

5. Find a basis for $\mathbf{Q}(\sqrt{5}, \sqrt[3]{5})$ over $\mathbf{Q}$.

   *Solution:* The set $\{1, \sqrt[3]{5}, \sqrt[3]{25}\}$ is a basis for $\mathbf{Q}(\sqrt[3]{5})$ over $\mathbf{Q}$, and since this extension has degree 3, the minimal polynomial $x^2 - 5$ of $\sqrt{5}$ remains irreducible in the extension $\mathbf{Q}(\sqrt[3]{5})$. Therefore $\{1, \sqrt{5}$ is a basis for $\mathbf{Q}(\sqrt{5}, \sqrt[3]{5})$ over $\mathbf{Q}(\sqrt[3]{5})$, and so the proof of Theorem 6.2.4 shows that the required basis is $\{1, \sqrt{5}, \sqrt[3]{5}, \sqrt{5}\sqrt[3]{5}, \sqrt[3]{25}, \sqrt{5}\sqrt[3]{25}\}$.

6. Show that $[\mathbf{Q}(\sqrt{2} + \sqrt[3]{5}) : \mathbf{Q}] = 6$.

   *Solution:* The set $\{1, \sqrt[3]{5}, \sqrt[3]{25}\}$ is a basis for $\mathbf{Q}(\sqrt[3]{5})$ over $\mathbf{Q}$, and since this extension has degree 3, the minimal polynomial $x^2 - 2$ of $\sqrt{2}$ remains irreducible over the extension $\mathbf{Q}(\sqrt[3]{5})$. Thus $\{1, \sqrt[3]{5}, \sqrt[3]{25}, \sqrt{2}, \sqrt{2}\sqrt[3]{5}, \sqrt{2}\sqrt[3]{25}\}$ is a basis for $\mathbf{Q}(\sqrt[3]{5}, \sqrt{2})$ over $\mathbf{Q}$, and this extension contains $u = \sqrt{2} + \sqrt[3]{5}$. It follows that $u$ has degree 2, 3, or 6 over $\mathbf{Q}$.

   We will show that $u$ cannot have degree $\leq 3$. If $\sqrt{2} + \sqrt[3]{5}$ is a root of a polynomial $ax^3 + bx^2 + cx + d$ in $\mathbf{Q}[x]$, then

   $a(\sqrt{2} + \sqrt[3]{5})^3 + b(\sqrt{2} + \sqrt[3]{5})^2 + c(\sqrt{2} + \sqrt[3]{5}) + d =$

   $a(2\sqrt{2} + 6\sqrt[3]{5} + 3\sqrt{2}\sqrt[3]{25} + 5) + b(2 + 2\sqrt{2}\sqrt[3]{5} + \sqrt[3]{25}) + c(\sqrt{2} + \sqrt[3]{5}) + d =$

   $(5a + 2b + d) \cdot 1 + (6a + c)\sqrt[3]{5} + b\sqrt[3]{25} + (2a + c)\sqrt{2} + 2b\sqrt{2}\sqrt[3]{5} + 3a\sqrt{2}\sqrt[3]{25} = 0.$

   Since $\{1, \sqrt[3]{5}, \sqrt[3]{25}, \sqrt{2}, \sqrt{2}\sqrt[3]{5}, \sqrt{2}\sqrt[3]{25}\}$ are linearly independent over $\mathbf{Q}$, it follows immediately that $a = b = 0$, and then $c = d = 0$ as well, so $\sqrt{2} + \sqrt[3]{5}$ cannot satisfy a nonzero polynomial of degree 1, 2, or 3 over $\mathbf{Q}$. We conclude that $[\mathbf{Q}(\sqrt{2} + \sqrt[3]{5}) : \mathbf{Q}] = 6$.

7. Find $[\mathbf{Q}(\sqrt[7]{16} + 3\sqrt[7]{8}) : \mathbf{Q}]$.

*Solution:* Let $u = \sqrt[7]{16} + 3\sqrt[7]{8}$. Since $u = (\sqrt[7]{2} + 3)(\sqrt[7]{2})^3$, it follows that $u \in \mathbf{Q}(\sqrt[7]{2})$. Since $x^7 - 2$ is irreducible over $\mathbf{Q}$ by Eisenstein's criterion, we have $[\mathbf{Q}(\sqrt[7]{2}) : \mathbf{Q}] = 7$, and then $u$ must have degree 7 over $\mathbf{Q}$ since $[\mathbf{Q}(u) : \mathbf{Q}]$ is a divisor of $[\mathbf{Q}(\sqrt[7]{2}) : \mathbf{Q}]$.

8. Find the degree of $\sqrt[3]{2} + i$ over $\mathbf{Q}$. Does $\sqrt[4]{2}$ belong to $\mathbf{Q}(\sqrt[3]{2} + i)$?

   *Solution:* Let $\alpha = \sqrt[3]{2} + i$, so that $\alpha - i = \sqrt[3]{2}$. Then $(\alpha - i)^3 = 2$, so we have $\alpha^3 - 3i\alpha^2 + 3i^2\alpha - i^3 = 2$, or $\alpha^3 - 3i\alpha^2 - 3\alpha + i = 2$. Solving for $i$ we get $i = (\alpha^3 - 3\alpha - 2)/(3\alpha^2 - 1)$, and this shows that $i \in \mathbf{Q}(\sqrt[3]{2} + i)$. It follows immediately that $\sqrt[3]{2} \in \mathbf{Q}(\sqrt[3]{2} + i)$, and so $\mathbf{Q}(\sqrt[3]{2} + i) = \mathbf{Q}(\sqrt[3]{2}, i)$.

   Since $x^3 - 2$ is irreducible over $\mathbf{Q}$, the number $\sqrt[3]{2}$ has degree 3 over $\mathbf{Q}$. Since $x^2 + 1$ is irreducible over $\mathbf{Q}$, we see that $i$ has degree 2 over $\mathbf{Q}$. Therefore $[\mathbf{Q}(\sqrt[3]{2} + i) : \mathbf{Q}] \leq 6$. On the other hand, $[\mathbf{Q}(\sqrt[3]{2} + i) : \mathbf{Q}] = [\mathbf{Q}(\sqrt[3]{2} + i) : \mathbf{Q}(\sqrt[3]{2})][\mathbf{Q}(\sqrt[3]{2}) : \mathbf{Q}]$ and $[\mathbf{Q}(\sqrt[3]{2} + i) : \mathbf{Q}] = [\mathbf{Q}(\sqrt[3]{2} + i) : \mathbf{Q}(i)][\mathbf{Q}(i) : \mathbf{Q}]$ so $[\mathbf{Q}(\sqrt[3]{2} + i) : \mathbf{Q}]$ must be divisible by 2 and 3. Therefore $[\mathbf{Q}(\sqrt[3]{2} + i) : \mathbf{Q}] = 6$.

   Finally, $\sqrt[4]{2}$ has degree 4 over $\mathbf{Q}$ since $x^4 - 2$ is irreducible over $\mathbf{Q}$, so it cannot belong to an extension of degree 6 since 4 is not a divisor of 6.