

PREFACE

An abstract algebra course at the junior/senior level, whether for one or two semesters, has been a well-established part of the curriculum for mathematics majors for over a generation. Our book is intended for this course, and has grown directly out of our experience in teaching the course at Northern Illinois University.

As a prerequisite to the abstract algebra course, our students are required to have taken a sophomore level course in linear algebra that is largely computational, although they have been introduced to proofs to some extent. Our classes include students preparing to teach high school, but almost no computer science or engineering students. We certainly do not assume that all of our students will go on to graduate school in pure mathematics.

In searching for appropriate text books, we have found several texts that start at about the same level as we do, but most of these stay at that level, and they do not teach nearly as much mathematics as we desire. On the other hand, there are several fine books that start and finish at the level of our Chapters 3 through 6, but these books tend to begin immediately with the abstract notion of group (or ring), and then leave the average student at the starting gate. We have in the past used such books, supplemented by our Chapter 1.

Historically the subject of abstract algebra arose from concrete problems, and it is our feeling that by beginning with such concrete problems we will be able to generate the student's interest in the subject and at the same time build on the foundation with which the student feels comfortable.

Although the book starts in a very concrete fashion, we increase the level of sophistication as the book progresses, and, by the end of Chapter 6, all of the topics taught in our course have been covered. It is our conviction that the level of sophistication should increase, slowly at first, as the students become familiar with the subject. We think our ordering of the topics speaks directly to this assertion.

Recently there has been a tendency to yield to demands of "relevancy," and to include "applications" in this course. It is our feeling that such inclusions often tend to be superficial. In order to make room for the inclusion of applications, some important mathematical concepts have to be sacrificed. It is clear that one must have substantial experience with abstract algebra before any genuine applications can be treated. For this reason we feel that the most honest introduction concentrates on the algebra. One of the reasons frequently given for treating applications is that they motivate the student. We prefer to motivate the subject with concrete problems from areas that the students have previously encountered, namely, the integers and polynomials over the real numbers.

One problem with most treatments of abstract algebra, whether they begin with group theory or ring theory, is that the students simultaneously encounter for the first time both abstract mathematics and the requirement that they produce proofs of their own devising. By taking a more concrete approach than is usual, we hope to separate these two initiations.

In three of the first four chapters of our book we discuss familiar concrete mathematics: number theory, functions and permutations, and polynomials. Although the objects of study are concrete, and most are familiar, we cover quite a few non-trivial ideas and at the same time introduce the student to the subtle ideas of mathematical proof. (At Northern Illinois University, this course and Advanced Calculus are the traditional places for students to learn how to write proofs.) After studying Chapters 1 and 2, the students have at their disposal some of the most important examples of groups—permutation groups, the group of integers modulo n , and certain matrix groups. In Chapter 3 the abstract definition of a group is introduced, and the students encounter the notion of a group armed with a variety of concrete examples.

Probably the most difficult notion in elementary group theory is that of a factor group. Again this is a case where the difficulty arises because there are, in fact, two new ideas encountered together. We have tried to separate these by treating the notions of equivalence relation and partition in Chapter 2 in the context of sets and functions. We consider there the concept of factoring a function into “better” functions, and show how the notion of a partition arises in this context. These ideas are related to the integers modulo n , studied in Chapter 1. When factor groups are introduced in Chapter 3, we have partitions and equivalence relations at our disposal, and we are able to concentrate on the group structure introduced on the equivalence classes.

In Chapter 4 we return to a more concrete subject when we derive some important properties of polynomials. Here we draw heavily on the students’ familiarity with polynomials from high school algebra and on the parallel between the properties of the integers studied in Chapter 1 and the polynomials. Chapter 5 then introduces the abstract definition of a ring after we have already encountered several important examples of rings: the integers, the integers modulo n , and the ring of polynomials with coefficients in any field.

From this point on our book looks more like a traditional abstract algebra textbook. After rings we consider fields, and we include a discussion of root adjunction as well as the three problems from antiquity: squaring the circle, duplicating the cube, and trisecting an angle. We also discuss splitting fields and finite fields here. We feel that the first six chapters represent the most that students at institutions such as ours can reasonably absorb in a year.

Chapter 7 returns to group theory to consider several more sophisticated ideas including those needed for Galois theory, which is the subject matter of Chapter 8. In Chapter 9 we return to a study of rings, and consider questions of unique factorization. As a number theoretic application, we present a proof of Fermat’s last theorem for the exponent 3. In fact, this is the last of a thread of number theoretic

applications that run through the text, including a proof of the quadratic reciprocity law in Section 6.7 and a study of primitive roots modulo p in Section 7.5. The applications to number theory provide topics suitable for honors students.

The last three chapters are intended to make the book suitable for an honors course or for classes of especially talented or well-prepared students. In these chapters the writing style is rather terse and demanding. Proofs are included for the Sylow theorems, the structure theorem for finite abelian groups, theorems on the simplicity of the alternating group and the special linear group over a finite field, the fundamental theorem of Galois theory, Abel's theorem on the insolvability of the quintic, and the theorem that a polynomial ring over a unique factorization domain is again a unique factorization domain.

The only prerequisite for our text is a sophomore level course in linear algebra. We do not assume that the student has been required to write, or even read, proofs before taking our course. We do use examples from matrix algebra in our discussion of group theory, and we draw on the computational techniques learned in the linear algebra course—see, for example, our treatment of the Euclidean algorithm in Chapter 1.

We have included a number of appendices to which the student may be referred for background material. The appendices on induction and on the complex numbers might be appropriate to cover in class, and so they include some exercises.

In our classes we usually intend to cover Chapters 1, 2 and 3 in the first semester, and most of Chapters 4, 5 and 6 in the second semester. In practice, we usually begin the second semester with group homomorphisms and factor groups, and end with geometric constructions. We have rarely had time to cover splitting fields and finite fields. For students with better preparation, Chapters 1 and 2 could be covered more quickly. The development is arranged so that Chapter 7 on the structure of groups can be covered immediately after Chapter 3. On the other hand, the material from Chapter 7 is not really needed until Section 8.4, at which point we need results on solvable groups.

We have included answers to some of the odd numbered computational exercises. In the exercise sets, the problems for which answers are given in the answer key are marked by the symbol †.

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To list all of the many sources from which we have learned is almost impossible. Perhaps because we are ring theorists ourselves, we have been attracted to and influenced by the work of two ring theorists—I. N. Herstein in *Topics in Algebra* and N. Jacobson in *Basic Algebra I, II*. In most cases our conventions, notation, and symbols are consistent with those used by Jacobson. We certainly need to mention the legacy of E. Noether, which we have met via the classic text *Algebra* by B. L. van der Waerden. Our treatment of Galois theory is influenced by the writing of E. Artin. In many ways our approach to abstract concepts via concrete

examples is similar in flavor to that of Birkhoff and Mac Lane in *A Survey of Modern Algebra*, although we have chosen to take a naive approach to the development of the number systems and have omitted any discussion of ordered fields. We have also been influenced by the historical approaches and choice of material in *Abstract Algebra: A First Course* by L. Goldstein and *Introduction to Abstract Algebra* by L. Shapiro.

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John A. Beachy
William D. Blair