Universal localization at semiprime Goldie ideals

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Background (commutative rings)

An ideal I of a ring R (with 1) is a subset of R such that $a, b \in I$ implies $a + b \in I$ and $a \in I$, $r \in R$ implies $ra \in I$ and $ar \in I$.

The ideals of the ring $\mathbb{Z}$ are the sets of multiples $n\mathbb{Z}$ of an integer n. Then m is a multiple of n iff $m \in n\mathbb{Z}$ iff $m\mathbb{Z} \subseteq n\mathbb{Z}$. Thus the lattice of ideals of $\mathbb{Z}$ is just as complicated as the relationship of divisibility of integers.

Note that $n > 0$ is a prime number if and only if $ab \in n\mathbb{Z}$ implies $a \in n\mathbb{Z}$ or $b \in n\mathbb{Z}$, for any integers a, b.

Look at the subring of the field $\mathbb{Q}$ of rational numbers defined by

$$\mathbb{Z}_{(2)} = \left\{ \frac{a}{c} \mid c \text{ is an odd integer} \right\} = \left\{ \frac{a}{c} \mid c \notin 2\mathbb{Z} \right\} .$$

It's a subring because the product of odd integers is again odd.

Background (commutative rings)

Let

$$M = \left\{ \frac{a}{c} \mid a \text{ is even and } c \text{ is odd} \right\} \subseteq \mathbb{Z}_{(2)}$$

be the ideal generated by 2. If $a/c \in \mathbb{Z}_{(2)}$ but $a/c \notin M$, then a is odd, so a/c is invertible in $\mathbb{Z}_{(2)}$. This shows that no element outside M generates a proper ideal, so M is a maximal ideal.

More generally, if p is a prime number, let

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{c} \mid c \notin p\mathbb{Z} \right\} .$$

This is a subring of $\mathbb{Q}$ since $\{c \in \mathbb{Z} \mid c \notin p\mathbb{Z}\}$ is a multiplicatively closed set. It can be shown that the ideals of $\mathbb{Z}_{(p)}$ are in one-to-one correspondence with the powers of p, so we have constructed a ring closely related to $\mathbb{Z}$ but with a much simpler ideal structure.

Background (commutative rings)

Let R be a commutative Noetherian ring and let $P \subseteq R$ is a prime ideal. (An ideal I of a commutative ring is prime if $ab \in I$ implies $a \in I$ or $b \in I$.) A ring R_P and ring homomorphism $\lambda : R \to R_P$ can be constructed for which every element of the complement $C(P)$ of P is inverted by λ.

The construction uses ordered pairs (c, a) (*think* $c^{-1}a$) where $c \in C(P)$ and $a \in R$, subject to the equivalence relation $(c, a) \sim (d, b)$ if there exists $c' \in C(P)$ with $c'(ad - bc) = 0$.

The ideal PR_P generated by P is the unique maximal ideal of R_P.

Motivation

David Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, p. 57:

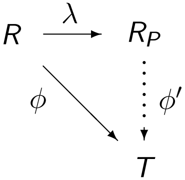
"A *local ring* is a ring with just one maximal ideal. Every since Krull's paper (1938) local rings have occupied a central position in commutative algebra. The technique of *localization* reduces many problems in commutative algebra to problems about local rings. This often turns out to be extremely useful.
Most of the problems with which commutative algebra has been successful are those that can be reduced to the local case."

Motivation

Properties of R_P:

(1). $\lambda : R \to R_P$ is universal with respect to the property that if $c \in C(P)$ then $\lambda(c)$ has an inverse in R_P.

That is, if $\phi : R \to T$ inverts $C(P)$, then there exists a unique ring homomorphism ϕ' such that the following diagram commutes.



Properties of R_P:

(2). The ideal PR_P is the unique maximal ideal of R, and R_P/PR_P is isomorphic to $Q(R/P)$, the quotient field of R/P.

(3). The functor $R_P \otimes_R \_ : R{-}\mathrm{Mod} \to R_P{-}\mathrm{Mod}$ preserves short exact sequences (i.e. R_P is a flat module)

(4). For any R-module M, the kernel of the mapping $M \to R_P \otimes_R M$ is the $C(P)$-torsion submodule $\{m \in M \mid cm = 0 \text{ for some } c \in C(P)\}$.

The noncommutative case

In a noncommutative ring R an ideal P is prime if $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$, for all ideals A, B of R.

Example 1. Let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ and $P = \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{bmatrix}$. Then P is prime since the ideals of R are in one-to-one correspondence with the ideals of the $\mathbb{Z}$, and the correspondence respects products.

Note that $R/P \cong \begin{bmatrix} \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} \end{bmatrix}$. This factor ring has divisors of zero, but at least it is a full matrix ring over a field.

The logical candidate for a localization of R at P is $\begin{bmatrix} \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \\ \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \end{bmatrix}$, which can be constructed by inverting all scalar matrices with an odd entry, thanks to the existence of adjoints.

The analog of the field of fractions of an integral domain

If R is a subring of Q, then R is a *left order* in Q if
(i) each regular element $c \in R$ (i.e. c is not a divisor of zero) has an inverse in Q, and
(ii) each $q \in Q$ can be written in the form $c^{-1}a$, for $a, c \in R$, where c is regular.

To put the product ac^{-1} into standard form, given a and c we need to be able to find a_1 and c_1 with

$$ac^{-1} = c_1^{-1}a_1 \,, \quad \text{or} \quad c_1 a = a_1 c \,.$$

Given this *left Ore condition*, $c^{-1}a \cdot d^{-1}b$ can be put into standard form by finding a_1 and d_1 with $ad^{-1} = d_1^{-1}a_1$, so that

$$\begin{aligned}(c^{-1}a)(d^{-1}b) &= c^{-1}(ad^{-1})b = c^{-1}(d_1^{-1}a_1)b \\ &= (c^{-1}d_1^{-1})(a_1 b) = (d_1 c)^{-1}(a_1 b).\end{aligned}$$

The analog of the field of fractions of an integral domain

Goldie's theorem (1958) shows that R is a left order in a full ring of $n \times n$ matrices over a skew field if and only if R is a prime ring with ascending chain condition on left annihilators and finite uniform dimension. (These finiteness conditions always hold when R is left Noetherian.) This ring of quotients is called the *classical ring of left quotients of R* and is denoted by $Q_{cl}(R)$.

We are now ready to look at noncommutative localization. We focus on prime ideals of R for which $Q_{cl}(R/P)$ exists. We would like to invert elements of $C(P)$, which we must now define as the set of elements that are regular modulo P (not as the complement of P). Equivalently, these are the elements inverted by the canonical homomorphism $R \to R/P \to Q_{cl}(R/P)$.

In Example 1, where P is the set of 2×2 matrices with even entries, $C(P)$ is the set of matrices whose determinant is odd.

Ore localization

If P is a prime Goldie ideal for which $C(P)$ satisfies the left Ore condition and is left reversible (if $ac = 0$ for $c \in C(P)$, then $c'a = 0$ for some $c' \in C(P)$) then the construction of a localization R_P goes through much as in the commutative case, and all four of the properties listed above still hold.

Even in very nice cases the left Ore condition may not hold.

Example 2.

$$R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}, P_1 = \begin{bmatrix} 2\mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}, P_2 = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 2\mathbb{Z} \end{bmatrix}.$$

Then $R/P_i \cong \mathbb{Z}/2\mathbb{Z}$ and so $Q_{cl}(R/P_i) = R/P_i$ is a field, making P_i as nice a prime Goldie ideal as possible.

Checking the Ore condition

The ideal P_1 satisfies the left Ore condition:
given $a = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \in R$ and $c = \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \in C(P_1)$ (i.e. c_{11} is odd) we need to solve $c'a = a'c$ with $c' \in C(P)$.
$\begin{bmatrix} c_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix}$.

The ideal P_2 does *not* satisfy the left Ore condition:
given $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in R$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in C(P_2)$ the equation
$\begin{bmatrix} c_{11} & 0 \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
requires $c_{22} = 0 = a_{22}$, so there is no solution with c_{22} odd.

Category theoretic localization

In the absence of the left Ore condition we could focus on the last two conditions that hold in the commutative case.

(3). The functor $R_P \otimes_R \_ : R{-}\mathrm{Mod} \to R_P{-}\mathrm{Mod}$ takes short exact sequences to short exact sequences.

(4). For any R-module M, the kernel of the mapping $M \to R_P \otimes_R M$ is the $C(P)$-torsion submodule $\{m \in M \mid cm = 0 \text{ for some } c \in C(P)\}$.

If the $C(P)$-torsion submodule is changed to $\{m \in M \mid \forall r \in R,\ crm = 0 \text{ for some } c \in C(P)\}$, it can be used to construct an exact functor into an abelian category consisting of "quotient" modules.

Category theoretic localization

There is a large body of work studying this method, beginning with Gabriel's thesis *Des catégories abéliennes*, Bull. Soc. Math. France **90** 323–448, published in 1962.

In the case of a prime Goldie ideal, this quotient functor does produce a ring $R_{C(P)}$, but properties (1) and (2) from the commutative case may be lost.

For the prime ideal P_2 in Example 2 the torsion-theoretic construction produces $R_{C(P_2)} = \begin{bmatrix} \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \\ \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \end{bmatrix}$, not a bad answer, but its unique maximal ideal is not closely connected to P_2.

We now turn to properties (1) and (2) of the commutative case:
(1). $\lambda : R \to R_P$ is universal with respect to the property that if $c \in C(P)$ then $\lambda(c)$ has an inverse in R_P.
(2). The ideal PR_P is the unique maximal ideal of R, and R_P/PR_P is isomorphic to $Q(R/P)$.

A ring satisfying (1) can be defined, but it may be the zero ring.

A new approach inverting matrices rather than elements was introduced by Cohn in **Free Rings and Their Relations** (1971) and *Inversive localization in Noetherian rings*, Commun. Pure Appl. Math. **26** (1973), 679-691.

It's convenient to generalize to a semiprime ideal S for which $Q_{cl}(R/S)$ exists and is semisimple Artinian, i.e. for which R/S is a semiprime left Goldie ring. In this case we say that S is a semiprime Goldie ideal.

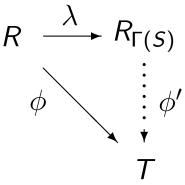
Definition of the universal localization

Let S be a semiprime Goldie ideal, and let $\Gamma(S)$ be the set of all square matrices inverted by the canonical mapping $R \to R/S \to Q_{cl}(R/S)$.

Definition (Cohn, 1973, Noetherian case)

The *universal localization* $R_{\Gamma(S)}$ of R at a semiprime Goldie ideal S is the ring universal with respect to inverting all matrices in $\Gamma(S)$.

That is, if $\phi : R \to T$ inverts all matrices in $\Gamma(S)$, then there exists a unique ring homomorphism ϕ' such that the following diagram commutes.



Elementary properties

The good: If S is left localizable, then $R_{\Gamma(S)} = R_S$.

Theorem (Cohn, 1971, Noetherian case)

Let S be a semiprime Goldie ideal of R.

(a) The universal localization of R at S exists.

(b) Each element of $R_{\Gamma(S)}$ is an entry in a matrix of the form $(\lambda(C))^{-1}$, for some $C \in \Gamma(S)$.

(c) The canonical mapping $\lambda : R \to R_{\Gamma(S)}$ is an epimorphism in the category of rings.

Theorem (1981)

The ring $R_{\Gamma(S)}$ is flat as a right module over R if and only if S is a left localizable ideal.

The bad: If $P_3 = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 3\mathbb{Z} \end{bmatrix}$, then $R_{\Gamma(P_1 \cap P_3)} = \begin{bmatrix} \mathbb{Z}_{(2)} & 0 \\ \mathbb{Q} & \mathbb{Z}_{(3)} \end{bmatrix}$.
Example 3: This universal localization is *not* left Noetherian.

The ugly: Cohn's construction showing that $R_{\Gamma(S)}$ exists:

For each n and each $n \times n$ matrix $[c_{ij}]$ in $\Gamma(S)$,

take a set of n^2 symbols $[d_{ij}]$,

and take a ring presentation of $R_{\Gamma(S)}$ consisting of all of the elements of R, as well as all of the elements d_{ij} as generators;

as defining relations take all of the relations holding in R,

together with all of the relations $[c_{ij}][d_{ij}] = I$ and $[d_{ij}][c_{ij}] = I$ which define all of the inverses of the matrices in $\Gamma(S)$.

It's a miracle that we get nothing more than what we wanted.

A bit of information about the kernel

Theorem

If ${}_RK \subseteq S$ is finitely generated, then $SK = K$ implies $K \subseteq \ker(\lambda)$.

Proof. Let $K = \sum_{i=1}^{n} Rx_i$, for $x_1, \ldots, x_n \in R$.
Since $K = SK$, we have $K = \sum_{i=1}^{n} Sx_i$. Let $\mathbf{x} = (x_1, \ldots, x_n)$. We can write $\mathbf{x}^t = C\mathbf{x}^t$, where the $n \times n$ matrix C has entries in S. Thus $(I_n - C)\mathbf{x}^t = \mathbf{0}^t$. But $I_n - C$ is invertible modulo S, so it certainly belongs to $\Gamma(S)$. Therefore the entries of $\mathbf{x}$ must belong to $\ker(\lambda)$, and so $K \subseteq \ker(\lambda)$.

Corollary

If P is idempotent, then $P = \ker(\lambda)$, and $R_{\Gamma(P)} = Q_{cl}(R/P)$, so $R_{\Gamma(P)}$ can be determined for any hereditary Noetherian prime ring.

Proof. In HNP rings a prime ideal is localizable or idempotent.

A characterization of $R_{\Gamma(S)}$

Let $J(R)$ be the intersection of the maximal left ideals of R.

Theorem (Cohn, 1973, Noetherian case)

$R_{\Gamma(S)}$ modulo its Jacobson radical is isomorphic to $Q_{cl}(R/S)$.

Theorem (1981)

For a semiprime Goldie ideal S, $R_{\Gamma(S)}$ is universal with respect to $R_{\Gamma(S)}/J(R_{\Gamma(S)}) \cong Q_{cl}(R/S)$.

More carefully, if $\phi : R \to T$ is a ring homomorphism such that $\overline{\phi} : R/P \to T/J(T)$ is the embedding of R/P in $Q_{cl}(R/P)$, then

$$\begin{array}{ccc} R & \xrightarrow{\lambda} & R_{\Gamma(S)} \\ & \underset{\phi}{\searrow} & \vdots\ \phi' \\ & & T \end{array}$$

A successful calculation

Theorem (1981)

Let R be left Noetherian, with N its prime radical (the intersection of all prime ideals), $\lambda : R \to R_{\Gamma(N)}$, $K = \ker(\lambda)$.

(a) The kernel K is the intersection of all ideals $I \subseteq N$ such that $C(N) \subseteq C(I)$.

(b) The ring R/K is a left order in a left Artinian ring, and $R_{\Gamma(N)}$ is naturally isomorphic to $Q_{cl}(R/K)$.

This theorem makes it possible to define an analog of the symbolic powers that are important in the commutative case. It is also the key to defining and using the reduced rank of the universal localization of a module. This provides a noncommutative tool useful in Stafford's generalization of the Forster-Swan theorem.

Another construction of the universal localization

Let S be a semiprime Goldie ideal of R. Each element in the universal localization $R_{\Gamma(S)}$ has the form $\lambda(e_i)\lambda(C)^{-1}\lambda(e_j)^t$ for some matrix $C \in \Gamma_n(S)$ and unit vectors e_i, e_j in R^n.

Let X be a left R-module. To construct a module of quotients $\Gamma^{-1}X$, instead of modeling elements of the form $c^{-1}x$, where $c \in C(S)$, and $x \in X$, we model elements of the form

$$\lambda(a)\lambda(C)^{-1}\mu(x)^t\,,$$

where $\mu : X \to \Gamma^{-1}X$. We consider ordered triples

$$(a, C, x^t)$$

where $a \in R^n$, $C \in \Gamma_n(S)$, and $x \in X^n$, for all positive integers n.

The first equivalence relation

Model: If C, U, V are matrices that are already invertible, then
$aC^{-1}x^t = a(UU^{-1})C^{-1}(V^{-1}V)x^t = aU(VCU)^{-1}Vx^t$.

Definition

$(aU, VCU, Vx^t) \equiv (a, C, x^t)$ if U, V are invertible matrices.

Write $(a : C : x^t)$ for the congruence classes.

Ultimately this congruence relation does not suffice since we cannot identify triples of different sizes.

Addition (avoiding the Ore condition)

Definition

$$(a : C : x^t) + (b : D : y^t) = \left(\begin{bmatrix} a & b \end{bmatrix} : \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} : \begin{bmatrix} x^t \\ y^t \end{bmatrix} \right)$$

Model for addition: Suppose C, D are already invertible.

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} x^t \\ y^t \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} C^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} x^t \\ y^t \end{bmatrix} =$$
$$\begin{bmatrix} aC^{-1} & bD^{-1} \end{bmatrix} \begin{bmatrix} x^t \\ y^t \end{bmatrix} = aC^{-1}x^t + bD^{-1}y^t$$

This is a commutative, associative binary operation.

Scalar multiplication (avoiding the Ore conditiion)

Definition

$$(a : C : r^t) \cdot (b : D : y^t) = \left([a \;\; 0] : \begin{bmatrix} C & -r^t b \\ 0 & D \end{bmatrix} : \begin{bmatrix} 0 \\ y^t \end{bmatrix} \right)$$

Model for scalar multiplication: Suppose C, D are invertible.

$$[a \quad 0] \begin{bmatrix} C & -r^t \cdot b \\ 0 & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ y^t \end{bmatrix} =$$

$$[a \quad 0] \begin{bmatrix} C^{-1} & C^{-1} r^t \cdot bD^{-1} \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ y^t \end{bmatrix} =$$

$$[aC^{-1} \quad aC^{-1} r^t \cdot bD^{-1}] \begin{bmatrix} 0 \\ y^t \end{bmatrix} = aC^{-1} r^t \cdot bD^{-1} y^t$$

Constructing a group

Model: Suppose C is already invertible.

We certainly must have $aC^{-1}0 = 0$ and $0C^{-1}x^t = 0$.

Definition

Let K be the subsemigroup generated by all congruence classes of the form $(0 : C : x^t)$ and $(a : C : 0^t)$. Then we define

$(a : C : x^t) \sim (b : D : y^t)$ if there exist $z_1, z_2 \in K$ with $(a : C : x^t) + z_1 = (b : D : y^t) + z_2$

Theorem

The equivalence relation $\sim$ defines a congruence, and modding out by it produces an abelian group.

The module of quotients

Theorem (1989)

(a) The above addition and multiplication define a ring of quotients $\Gamma^{-1}R$ *and a module of quotients* $\Gamma^{-1}X$.

(b) Each element of $\Gamma^{-1}R$ *is an entry in the inverse of a matrix in* $\Gamma(S)$.

Theorem (1989)

$\Gamma^{-1}R \cong R_{\Gamma(S)}$ *and* $\Gamma^{-1}X \cong R_{\Gamma(S)} \otimes_R X$.

Back to the Ore condition?

Model:

If C, C_1 are invertible and $C_1 A = A_1 C$, then $AC^{-1} = C_1^{-1} A_1$, so $aAC^{-1}x^t = aC_1^{-1}A_1x^t$. Note: C and C_1 can have different sizes.

Theorem

If there exist $C_1 \in \Gamma(S)$ and A_1 (of the appropriate size) such that $C_1 A = A_1 C$, then

$$(aA : C : x^t) \sim (a : C_1 : A_1 x^t) \ .$$

This theorem is the key to simpler proofs of the results on $\Gamma^{-1}X$. In fact, it can be used to replace both of the previous equivalence relations.

End of the presentation

There are more slides with open questions.

Questions and future directions

(1) In the commutative Noetherian case the kernel of $\lambda : R \to R_P$ is the intersection of the symbolic powers of P. Are there conditions under which this is true in the noncommutative case? This is related to the question of when the universal localization coincides with Goldie's localization.

(2) Is there a broad class of rings for which the universal localization is well-behaved? Christine Leroux and I have some results for Noetherian rings finite over their center.

(3) Are there any chain conditions on left ideals that are preserved? This is related to work with Abby Bailey on piecewise Noetherian rings.

(4) Mauricio Medina and I have some results related to Goldie's notion of reduced rank. Hopefully there will be some applications to the study of Noetherian rings.

(5) The commutative Noetherian case gives rise to a sheaf of local rings. Is there a noncommutative analog?

Question 1: Symbolic powers

Definition

Let S be a semiprime Goldie ideal of R, with $\lambda : R \to R_{\Gamma(S)}$. The *$n^{th}$ symbolic power of S* is $S^{(n)} = \lambda^{-1}(R_{\Gamma(S)}\lambda(S^n)R_{\Gamma(S)})$.

Theorem

If R is left Noetherian, then the following conditions hold for the symbolic powers of the semiprime ideal S.

(a) $S^{(n)}$ is the intersection of all ideals I such that $S^n \subseteq I \subseteq S$ and $C(S) \subseteq C(I)$.

(b) $C(S)$ is a left Ore set modulo $S^{(n)}$.

(c) $R_{\Gamma(S)}\lambda(S^n)R_{\Gamma(S)} = (J(R_{\Gamma(S)})^n$, for all $n > 0$.

(d) $R/S^{(n)}$ is an order in the left Artinian ring $R_{\Gamma(S)}/(J(R_{\Gamma(S)}))^n$.

Question 1: Goldie's localization

In two papers in 1967 and and 1968, Goldie defined a localization at a prime ideal P of a Noetherian ring R by first factoring out the intersection $\cap_{n=1}^{\infty} P^{(n)}$ of the symbolic powers. He then took the inverse limit of the Artinian quotient rings $Q_{cl}(R/P^{(n)})$, and finally defined an appropriate subring of this inverse limit.

Theorem (1984)

Let P be a prime ideal of the Noetherian ring R. Then Goldie's localization of R at P is isomorphic to $R_{\Gamma(P)}/\bigcap_{n=1}^{\infty} J^n$, where J is the Jacobson radical of $R_{\Gamma(P)}$.

Question 2: Example 2 again

Example 2: $R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$, $P_2 = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 2\mathbb{Z} \end{bmatrix}$, $K = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & 0 \end{bmatrix}$.
Then $K^2 = K$, so $P_2 K = K$, and therefore $K \subseteq \ker\lambda$, for $\lambda : R \to R_{\Gamma(P_2)}$. It follows easily that $K = \ker\lambda$ and $R_{\Gamma(P_2)}$ is isomorphic to $\mathbb{Z}_{(2)}$.

An alternate approach: Recalling that $P_1 = \begin{bmatrix} 2\mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$, we can invert the scalar matrices in $C(P_1 \cap P_2)$ to obtain $R_{P_1 \cap P_2}$ $= \begin{bmatrix} \mathbb{Z}_{(2)} & 0 \\ \mathbb{Z}_{(2)} & \mathbb{Z}_{(2)} \end{bmatrix}$ with maximal ideal $\widehat{P_2} = \begin{bmatrix} \mathbb{Z}_{(2)} & 0 \\ \mathbb{Z}_{(2)} & 2\mathbb{Z}_{(2)} \end{bmatrix}$.
Factoring out $\cap_{i=n}^{\infty} \widehat{P_2}^n$ yields $R_{\Gamma(P_2)} \cong \mathbb{Z}_{(2)}$.

This illustrates a two-step approach: use the Ore localization at a suitable semiprime ideal, followed by its universal localization, which in this case is just a factor ring.

Question 2: Chain conditions on $R_{\Gamma(S)}$

Recall Example 3, which produced $R_{\Gamma(P_1 \cap P_3)} = \begin{bmatrix} \mathbb{Z}_{(2)} & 0 \\ \mathbb{Q} & \mathbb{Z}_{(3)} \end{bmatrix}$, which is no longer Noetherian. Bill Blair and I had to work much harder to produce such an example for a prime ideal. We ultimately found a complicated example, in which the ring is finitely generated as a module over its center.

Recall that in a ring finitely generated as a module over its center, the *clique* of a prime ideal P is the set of prime ideals with the same intersection down to the center of the ring.

Theorem (2016, with Christine Leroux)

If R is finitely generated as a module over its Noetherian center, and P is a prime ideal that does not contain the intersection of symbolic powers of any other prime ideal in the clique of P, then $R_{\Gamma(P)}$ is the homomorphic image of the Ore localization at the clique of P, and therefore it is Noetherian.

Question 3: Piecewise Noetherian rings

Let N be the prime radical of R. A left R/N-module M is said to have finite reduced rank if the module $Q_{cl}(R/N) \otimes_R M$ has finite length as a module over the Artinian ring $Q_{cl}(R/N)$. This definition can be extended to finitely generated modules, provided N is nilpotent.

A ring is said to be piecewise Noetherian if it has ACC on prime ideals and every factor ring has finite reduced rank.

The examples in this presentation are all piecewise Noetherian.

Open question: Is the universal localization of a piecewise Noetherian ring still piecewise Noetherian?

Recall that a commutative ring R is piecewise Noetherian if (i) R has Noetherian spectrum and (ii) for each ideal I and each prime ideal P minimal over I, the localized ring R_P/IR_P is Artinian.

Theorem

If R is left piecewise Noetherian, and P is a prime ideal minimal over the ideal I, then $R_{\Gamma(P)}/I^e R_{\Gamma(P)} \cong (R/I)_{\Gamma(P/I)}$ is a left Artinian ring.

Can we characterize left piecewise Noetherian rings via universal localization? For the prime radical N of R, and a module ${}_RM$, we have the following exact sequences:

$$0 \to J(R_{\Gamma(N)}) \to R_{\Gamma(N)} \to Q_{cl}(R/N) \to 0 \quad \text{as right } R\text{-modules}$$

$$0 \to NM \to M \to M/NM \to 0 \quad \text{as left } R\text{-modules}$$

Question 4: After tensoring

$$\begin{array}{ccccccc}
J(R_{\Gamma(N)}) \otimes_R NM & \longrightarrow & R_{\Gamma(N)} \otimes_R NM & \longrightarrow & Q_{cl}(R/N) \otimes_R NM & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
J(R_{\Gamma(N)}) \otimes_R M & \longrightarrow & R_{\Gamma(N)} \otimes_R M & \longrightarrow & Q_{cl}(R/N) \otimes_R M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
J(R_{\Gamma(N)}) \otimes_R M/NM & \longrightarrow & R_{\Gamma(N)} \otimes_R M/NM & \longrightarrow & Q_{cl}(R/N) \otimes_R M/NM & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}$$

Question 4: Reductions

Theorem

Let R be a left piecewise Noetherian ring with prime radical N. If M is a left R-module, then the Goldie rank (or reduced rank) $\rho(M/NM)$ of M/NM is given by the length of the module $\left(R_{\Gamma(N)} \otimes_R M\right)/J(R_{\Gamma(N)})\left(R_{\Gamma(N)} \otimes_R M\right)$.

Proof: In the diagram we have the following:
(1) $Q_{cl}(R/N)$ is a right R/N module, so it is annihilated by N, and therefore $Q_{cl}(R/N) \otimes_R NM = 0$.

(2) Since M/NM is a left R/N-module,
$Q_{cl}(R/N) \otimes_R M/NM = Q_{cl}(R/N) \otimes_{R/N} M/NM$.

(3) The image of the mapping from $J(R_{\Gamma(N)}) \otimes_R M$ into $R_{\Gamma(N)} \otimes_R M$ is $J(R_{\Gamma(N)})\left(R_{\Gamma(N)} \otimes_R M\right)$.

Question 4: The diagram for the proof of the theorem

$$
\begin{array}{ccccccc}
J(R_{\Gamma(N)}) \otimes_R NM & \longrightarrow & R_{\Gamma(N)} \otimes_R NM & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
J(R_{\Gamma(N)}) \otimes_R M & \longrightarrow & R_{\Gamma(N)} \otimes_R M & \longrightarrow & Q_{cl}(R/N) \otimes_R M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \cong & & \downarrow \\
J(R_{\Gamma(N)}) \otimes_R M/NM & \longrightarrow & R_{\Gamma(N)} \otimes_R M/NM & \longrightarrow & Q_{cl}(R/N) \otimes_{R/N} M/NM & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}
$$

Conclusion:

$$\left(R_{\Gamma(N)} \otimes_R M\right) / J(R_{\Gamma(N)}) \left(R_{\Gamma(N)} \otimes_R M\right) \cong Q_{cl}(R/N) \otimes_{R/N} M/NM$$

Example 4.

Let $R = \begin{bmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ and $P = \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$.

Then R is Noetherian and $P^2 = P$, so $P = \ker(\lambda)$ and $R_{\Gamma(P)} = R/P \cong \mathbb{Z}/2\mathbb{Z}$. Note that $R = \{r \in M_2(\mathbb{Z}) \mid rP \subseteq P\}$ is the idealizer of P in $M_2(R)$, since $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix} \subseteq \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ only requires $b \in 2\mathbb{Z}$.

Example 5.5.11 of McConnell/Robson implies that R is an hereditary Noetherian prime ring. A result of Chatters and Ginn shows that any prime ideal of such a ring is either idempotent or classically localizable, so in either case the universal localization can be determined.

Question 5: $\Gamma(P)$ vis-a-vis $\Gamma(0)$

In the commutative case, for prime ideals $P \subset Q$ we always have $C(P) \supset C(Q)$.

Recall that $R = \begin{bmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ and $P = \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$.

$\Gamma(P)$ is the set of matrices in R whose upper left hand entry is odd, since $R/P = Q_{cl}(R/P) \cong \mathbb{Z}/2\mathbb{Z}$.

R is easily checked to be a left order in $M_2(\mathbb{Q})$, showing again that (0) is a prime ideal of R, and that $\Gamma(0)$ is the set of matrices in R that are invertible in $M_2(\mathbb{Q})$, i.e. those with nonzero determinant.

An unfortunate conclusion: we have $(0) \subset P$, but the desired inclusion $\Gamma(0) \supset \Gamma(P)$ fails to hold.

Question 5: The sheaf of local rings

If R is a commutative Noetherian ring, the set of prime ideals has a topology whose closed sets correspond to semiprime ideals S, and consist of all prime ideals P with $S \subseteq P$ (equivalently, $C(S) \supseteq C(P)$). There is a natural homomorphism $R_P \to R_S$.

In the noncommutative case, for each semiprime ideal S define a closed subset consisting of those prime ideals P with $\Gamma(S) \supseteq \Gamma(P)$. Since $R_{\Gamma(P)}$ is universal with respect to inverting $\Gamma(P)$, there is still a natural homomorphism $R_{\Gamma(P)} \to R_{\Gamma(S)}$.

Question: Can any of the results from the commutative case be extended to the noncommutative case?

Some future directions for research

(A) Universal localization of enveloping algebras. McConnell [1968] obtained some results on Goldie's localization for certain enveloping algebras.

(B) Universal localization of group rings. Some topologists have been interested in universal localization at the augmentation ideal .

(C) Universal localization of additive categories. I think it may be possible to give a new construction of the derived category.