MATH 211 C EXAM II SOLUTIONS 10/18/2002
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1. (30 points) Find these derivatives (DO NOT SIMPLIFY YOUR ANSWERS).

(a) (6 pts) $f(x) = (x^4 - 3x^2 + 2x + 1)^8$ $\qquad f'(x) = 8(x^4 - 3x^2 + 2x + 1)^7(4x^3 - 6x + 2)$

(b) (6 pts) $f(x) = (x^2 + 1)^3(x^2 - 1)^2$ $\qquad f'(x) = 3(x^2 + 1)^2(2x)(x^2 - 1)^2 + (x^2 + 1)^3(2)(x^2 - 1)(2x)$

(c) (p163 #33; 6 pts) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ Using the quotient rule: $f'(x) = \dfrac{(2x)(x^2 + 1) - (x^2 - 1)(2x)}{(x^2 + 1)^2}$

Product rule: $f(x) = (x^2 - 1)(x^2 + 1)^{-1}$ $\qquad f'(x) = (2x)(x^2 + 1)^{-1} + (x^2 - 1)(-1)(x^2 + 1)^{-2}(2x)$

(d) (p163 #37; 6pts) $f(x) = \dfrac{4x + 3}{\sqrt{x}}$ $\qquad f(x) = \dfrac{4x}{\sqrt{x}} + \dfrac{3}{\sqrt{x}} = 4x^{1/2} + 3x^{-1/2}$ $\qquad f'(x) = 2x^{-1/2} - \frac{3}{2}x^{-3/2}$

Of course, you can use the quotient rule or product rule, as in (c), but if you can make an algebraic simplification it is almost always worth the time.

(e) (p172 #35; 6pts) $f(x) = \sqrt{\dfrac{x^2 + 1}{x^2 - 1}}$ $\qquad f(x) = \dfrac{\sqrt{x^2 + 1}}{\sqrt{x^2 - 1}} = (x^2 + 1)^{1/2}(x^2 - 1)^{-1/2}$

$f'(x) = \frac{1}{2}(x^2 + 1)^{-1/2}(2x)(x^2 - 1)^{-1/2} + (x^2 + 1)^{1/2}(-\frac{1}{2})(x^2 - 1)^{-3/2}(2x)$

2. (15 pts; p200 #23) The graph of $f(x) = \dfrac{-8}{x^2 + 1}$ has one relative extreme point. Find the coordinates of this point, and use the sign of $f'(x)$ to determine whether the point is a relative maximum or a relative minimum. *You do not need to include a sketch of the graph.*

Using the quotient rule (remember that the derivative of a constant is zero) you should get $f'(x) = \dfrac{(0)(x^2 + 1) - (-8)(2x)}{(x^2 + 1)^2} = \dfrac{16x}{(x^2 + 1)^2}$. Since the denominator is always positive, the sign of $f'(x)$ just depends on the sign of x, so $f'(x)$ is negative when $x < 0$ and positive when $0 < x$. This means that $(0, 0)$ is a relative minimum point.

If you decide to use the product rule, $f(x) = -8(x^2 + 1)^{-1}$, and then $f'(x) = (8)(x^2 + 1)^{-1}(2x)$, which gives the same answer as we got before.

If you set $f'(x) = 0$, you get $\dfrac{16x}{(x^2 + 1)^2} = 0$. Then you can solve the equation in two different ways.
You could say that a fraction is zero only when the numerator is zero, so you get $16x = 0$, or $x = 0$.
Or, since $(x^2 + 1)^2$ is nonzero for all x, you could multiply both sides of the equation by $(x^2 + 1)^2$.
$\dfrac{16x}{(x^2 + 1)^2} \cdot (x^2 + 1)^2 = 0 \cdot (x^2 + 1)^2$ gives the same answer: $16x = 0$, and then $x = 0$.

3. (10 pts; p155 #3) Given the cost function $C(x) = 0.001x^2 + 1.2x + 60$ and revenue function $R(x) = 5x$, find each of the following: (a) (3 pts) the profit function $P(x)$; (b) (3 pts) $R(100)$; $C(100)$; $P(100)$ (c) (4 pts) the marginal profit when $x = 100$.

To answer (d), remember that the marginal profit is given by the derivative of the profit function.

$P(x) = R(x) - C(x) = 5x - (0.001x^2 + 1.2x + 60) = 5x - 0.001x^2 - 1.2x - 60 = 3.8x - 0.001x^2 - 60$

$R(100) = 5(100) = 500$
$C(100) = (.001)(100)^2 + (1.2)(100) + 60 = (.001)(10000) + 120 + 60 = 10 + 180 = 190$
$P(100) = R(100) - C(100) = 500 - 190 = 310$

$P'(x) = 3.8 - .002x$ $\qquad P'(100) = 3.8 - (.002)(100) = 3.8 - .2 = 3.6$

4. (20 pts) Sketch the graph (5 pts) of the function $f(x) = x^3 - 3x^2 - 1$. First find:

(a) (4 pts) $f'(x)$ and $f''(x)$ (factor your answer);

$f'(x) = 3x^2 - 6x = 3x(x-2)$ $\qquad$ $f''(x) = 6x - 6 = 6(x-1)$

(b) (4 pts) critical points (if any); where $f(x)$ is increasing; where $f(x)$ is decreasing;

Setting $f'(x) = 0$ we get $x = 0$ or $x = 2$.

We need to check the sign of $f'(x)$ in three intervals: for $x < 0$; for $0 < x < 2$; and for $2 < x$. We can choose the points $x = -1$, $x = 1$, and $x = 3$ in these intervals. Then $f(-1) = 9$, $f'(1) = -3$, and $f'(3) = 9$, so we can conclude that

$f(x)$ is increasing for $x < 0$ $\quad$ $f(x)$ is decreasing for $0 < x < 2$ $\quad$ $f(x)$ is increasing for $2 < x < 0$

(c) (3 pts) inflection points (if any); where $f(x)$ is concave up; where $f(x)$ is concave down;

Setting $f''(x) = 0$ we get $x = 1$, and this gives an inflection point of $(1, -3)$ since $f''(x)$ changes sign. The graph is concave down for $x < 1$, and concave up for $1 < x$.

(d) (4 pts) the extreme points (relative maximum or relative minimum) of the curve (if any).

For the critical points $(0, -1)$ and $(2, -5)$, the curve is concave down at the first one since $f''(0)$ is negative, and concave up at the second since $f''(2)$ is positive. We conclude that $x = 0$ produces a relative maximum, while $x = 2$ gives a relative minimum.

5. (10 pts) Find an equation for the line tangent to the graph of $y = x(x-1)^5$ at the point $(2, 2)$.

Using the product rule, $y' = (x-1)^5 + (x)(5)(x-1)^4$. Substituting $x = 2$ gives the slope of the graph at $(2, 2)$: $m = (2-1)^5 + (2)(5)(2-1)^4 = 1^5 + (10)(1^4) = 1 + 10 = 11$. Answer: $\quad y = 11(x-2) + 2$

6. (15 pts; p232 #41) For the function $f(x) = \dfrac{2x^2}{x^2 - 16}$, find

(a) (4 pts) $f'(x) = \dfrac{4x(x^2-16) - 2x^2(2x)}{(x^2-16)^2} = \dfrac{4x^3 - 64x - 4x^3}{(x^2-16)^2} = \dfrac{-64x}{(x^2-16)^2}$

(b) (4 pts) critical points (if any); where the graph is increasing; where the graph is decreasing;

The only way a fraction can be equal to zero is if the numerator is zero.
Setting $f'(x) = 0$ gives $-64x = 0$, so $x = 0$ is the only critical point.
Then it is easy to see that $f'(x)$ is positive for $x < 0$ and negative for $0 < x$.
Conclusion: $f(x)$ is increasing for $x < 0$ and decreasing for $0 < x$.

(c) (4 pts) the vertical and horizontal asymptotes.

Vertical asymptotes: setting the denominator equal to zero gives $x = 4$ and $x = -4$.

Horizontal asymptote: $y = 2$ since $\displaystyle\lim_{x\to\infty} \frac{2x^2}{x^2-16} = \lim_{x\to\infty} \frac{2}{1 - 16/x^2} = 2$.

(d) (3 pts) Given $f''(x) = \dfrac{192x^2 + 1024}{(x^2-16)^3}$, find where the graph is concave up; concave down.

The numerator is positive for all values of x, so $f''(x)$ can only change sign in its denominator. When $-4 < x < 4$, the denominator is negative, and so $f''(x)$ is negative, and $f(x)$ is concave down in this interval. The graph is concave up for $x < -4$ and for $4 < x$, since on these intervals we have $x^2 - 16 > 0$, and therefore $f''(x)$ is positive.

The class average on the test was 69.4.

Grading scale: 90–100 A (12); 80–89 B (16); 65–79 C (26); 55–64 D (15); 22–54 F (13)