The grading curve on this exam was: A 170 (85%); B 150 (75%); C 120 (60%); D 100 (50%).

1. (30 points) Find the derivative of each of these functions. You do not need to simplify your answers.

(a) $f(x) = \dfrac{x^4}{4} - \sqrt[4]{x} - \dfrac{1}{x}$ *rewrite as* $f(x) = \dfrac{1}{4}x^4 - x^{1/4} - x^{-1}$ p148#19,29

$f'(x) = \dfrac{1}{4} \cdot 4x^3 - \dfrac{1}{4}x^{-3/4} - (-1)x^{-2} = x^3 - \dfrac{x^{-3/4}}{4} + \dfrac{1}{x^2}$

(b) $f(x) = (x^3 - 4x)^2$ $f'(x) = 2(x^3 - 4x)\dfrac{d}{dx}(x^3 - 4x) = 2(x^3 - 4x)(3x^2 - 4)$ p163#15

(c) $f(x) = x^2 \ln x - \dfrac{1}{2\,x^2}$ *rewrite as* $f(x) = x^2 \ln x - \dfrac{1}{2}x^{-2}$ p314#45

$f'(x) = 2x \ln x + x^2 \cdot \dfrac{1}{x} - \dfrac{1}{2}(-2)x^{-3} = 2x \ln x + x + \dfrac{1}{x^3}$

(d) $f(x) = \dfrac{xe^{-x}}{1 + x^2}$ $f'(x) = \dfrac{(e^{-x} - xe^{-x})(1 + x^2) - xe^{-x}(2x)}{(1 + x^2)^2}$ p299#66

2. (15 pts) Find the following limits algebraically.

(a) $\lim_{x \to -5} \dfrac{x^2 - 5}{x + 5} =$ The limit does not exist.

(b) $\lim_{x \to 2} \dfrac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} \dfrac{(x - 2)(x + 3)}{(x - 2)(x + 2)} = \lim_{x \to 2} \dfrac{x + 3}{x + 2} = \dfrac{5}{4}$ p113#21

3. (20 pts) Find the relative maximum and relative minimum points of $f(x) = -x^3 + 3x - 2$. p215#12

$f'(x) = -3x^2 + 3 = -3(x^2 - 1)$ and setting $f'(x) = 0$ we get $x = \pm 1$. Since $f''(x) = -6x$, we have $f''(1) = -6$ and $f''(-1) = 6$, which shows that $(1, 0)$ is a relative max and $(-1, -4)$ is a relative min.

Graph the function $f(x)$, using your knowledge of calculus.

The graph decreases through $(-2, 0)$, to the relative minimum at $(-1, -4)$, then increases to the relative maximum at $(1, 0)$, passing through $(0, -2)$, which is a point of inflection. Then the graph decreases, passing through $(2, -4)$.

4. (30 pts) Find the following integrals.

(a) $\displaystyle\int (x^4 - x^{6/5})\, dx = \frac{x^5}{5} - \frac{x^{11/5}}{11/5} + C = \frac{x^5}{5} - \frac{5x^{11/5}}{11} + C$ p371#14

(b) $\displaystyle\int_1^e (x + \frac{1}{x})\, dx = \frac{x^2}{2} + \ln|x| \Big|_1^e = (\frac{e^2}{2} + \ln e) - (\frac{1^2}{2} + \ln 1) = \frac{e^2}{2} + 1 - \frac{1}{2} - 0 = \frac{e^2}{2} + \frac{1}{2}$ p385#41

(c) $\displaystyle\int x^2\sqrt{x^3 + 1}\, dx = \int u^{1/2} \cdot \frac{1}{3}\, du = \frac{1}{3}\frac{u^{3/2}}{3/2} + C = \frac{2}{9}(x^3 + 1)^{3/2} + C$ p412#69

(Use the substitution $u = x^3 + 1$, so that $du = 3x^2 dx$, and then $x^2 dx = \dfrac{1}{3}du$.)

5. (15 pts) Find the area bounded by the curves $y = x^2$ and $y = x + 2$. (First sketch the graphs.) p406#3

The line $y = x + 2$ intersects the parabola $y = x^2$ at $(-1, 1)$ and $(2, 4)$. Since the line lies above the parabola on the interval $[-1, 2]$, the area is given by the integral

$$\int_{-1}^{2} [(x+2) - x^2]\, dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}\Big|_{-1}^{2} = (2+4-\frac{8}{3}) - (\frac{1}{2} - 2 + \frac{1}{3}) = 8 - \frac{8}{3} - \frac{1}{2} - \frac{1}{3} = 8 - 3 - \frac{1}{2} = 4\tfrac{1}{2} = \frac{9}{2}$$

6. (15 pts) An appliance firm is marketing a new refrigerator. It determines that in order to sell x refrigerators, the price per refrigerator must be $p = 280 - 0.4x$. It also determines that the total cost of producing x refrigerators is given by $C(x) = 5000 + 0.6x^2$. What price per refrigerator must be charged in order to make a maximum profit? p262#28

Profit: $P(x) = xp - C(x) = 280x - .4x^2 - (5000 + .6x^2) = 280x - x^2 - 5000$ $\quad P'(x) = 280 - 2x$
Setting $P'(x) = 0$ gives $x = 140$. Check: $P''(x) = -2$, so $x = 140$ maximizes the profit.

7. (15 pts) From a thin piece of cardboard 20 in. by 20 in. square corners are cut out so that the sides can be folded up to make a box. What dimensions will yield a box of maximum volume? p262#18

Let x be the width of the square corner that is cut out. Then the volume of the box is
$V(x) = (20 - 2x)(20 - 2x)(x) = (400 - 80x + 4x^2)x = 4x^3 - 80x^2 + 400x$.
$V'(x) = 12x^2 - 160x + 400 = 4(3x^2 - 40x + 100) = 4(x - 10)(3x - 10)$.
Setting $V'(x) = 0$ gives $x = 10$ or $x = 10/3$. The first solution is impossible (leaving nothing to fold up).
$V''(x) = 24x - 160$, so $V''(10/3) = 800 - 160 > 0$ and $V(x)$ is concave up at $x = 10/3$, showing that $x = 10/3$ gives a maximum volume. Answer: $10/3$ by $40/3$ by $40/3$ (since $20 - 2(10/3) = 40/3$)

8. (15 pts) The population of a colony of bacteria after t hours is given by $P(t) = 5000e^{0.02t}$.
(a) Find the rate of growth of the population after 50 hours.

$P'(t) = (.02)(5000)e^{0.02t} = 100e^{.02t}$ $\quad P'(50) = 100e^{.02(50)} = 100e$

(b) In how many hours will the population double?

Solve $P(t) = 10000$ $\quad 10000 = 5000e^{.02t}$ $\quad 2 = e^{.02t}$ $\quad \ln 2 = .02t$ $\quad t = \dfrac{\ln 2}{.02}$

9. (15 pts) Find y'' if $y = x\sqrt{1 + x^2}$. *Rewrite as:* $y = x(1 + x^2)^{1/2}$ p177#41

$y' = (1 + x^2)^{1/2} + x(\tfrac{1}{2})(1 + x^2)^{-1/2}(2x) = (1 + x^2)^{1/2} + x^2(1 + x^2)^{-1/2}$

$y'' = (\tfrac{1}{2})(1 + x^2)^{1/2}(2x) + 2x(1 + x^2)^{-1/2} + x^2(-\tfrac{1}{2})(1 + x^2)^{-3/2}(2x) = 3x(1 + x^2)^{-1/2} - x^3(1 + x^2)^{-3/2}$

10. (10 pts) A company determines that the marginal cost of producing the xth unit of a certain product is given by $C'(x) = x^3 - x$. Find the total cost $C(x)$, assuming fixed costs to be \$200. p371#32

The antiderivative of $C'(x)$ is $C(x) = \dfrac{x^4}{4} - \dfrac{x^2}{2} + K$. When $x = 0$, $C(x) = 200$, so $K = 200$.

11. (10 pts) $\dfrac{d}{dx} \ln\left[\dfrac{x^5}{(8x+5)^2}\right] = \dfrac{d}{dx}\left[\ln x^5 - \ln(8x+5)^2\right] = \dfrac{d}{dx}\left[5\ln x - 2\ln(8x+5)\right]$ p316#83

$= 5 \cdot \dfrac{1}{x} - 2 \cdot \dfrac{8}{8x+5} = \dfrac{5}{x} - \dfrac{16}{8x+5}$

12. (10 pts) Using the limit definition of the derivative, find the derivative of $f(x) = \dfrac{1}{x^2}$. p138#29

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2}\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x^2 - x^2 - 2xh - h^2}{x^2(x+h)^2}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{-2xh - h^2}{x^2(x+h)^2}\right] = \lim_{h\to 0} \frac{h(-2x-h)}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2}$$

$$= \frac{-2x}{x^2x^2} = \frac{-2}{x^3}$$