MATH 211 FINAL EXAM SOLUTIONS NAME ______________________________

5/6/2003 NO CALCULATORS! Show all necessary work. Be neat, clear, and brief.

1. (40 points) Find the derivative of each of these functions. You do not need to simplify your answers.

(a) $f(x) = 4\sqrt{x} + x - \dfrac{5}{x}$ $\quad f'(x) = 4 \cdot \frac{1}{2}x^{-1/2} + 1 + 5x^{-2} = \dfrac{2}{\sqrt{x}} + 1 + \dfrac{5}{x^2}$ p148#10,23

(b) $f(x) = (\ln x)^2$ $\quad f'(x) = 2(\ln x) \cdot \dfrac{1}{x}$ p315#61

(c) $f(x) = \dfrac{e^x}{x^2+1}$ $\quad f'(x) = \dfrac{e^x \cdot (x^2+1) - e^x \cdot (2x)}{(x^2+1)^2}$ p299#61

(d) $f(x) = \ln\left[\sqrt{5+x^2}\,\right] = \ln(5+x^2)^{1/2} = \dfrac{1}{2} \cdot \ln(5+x^2)$ $\quad f'(x) = \dfrac{1}{2} \cdot \dfrac{2x}{5+x^2} = \dfrac{x}{5+x^2}$ p316#84

2. (30 pts) Find the following integrals.

(a) $\displaystyle\int \left(x^3 + \frac{1}{x^3} - x^{8/7}\right)\,dx = \int \left(x^3 + x^{-3} - x^{8/7}\right)\,dx = \frac{x^4}{4} + \frac{x^{-2}}{-2} - \frac{x^{11/7}}{11/7} + C$ p371#13,18

(b) $\displaystyle\int (1-t)\sqrt{t}\,dt = \int (t^{1/2} - t^{3/2})\,dt = \frac{t^{3/2}}{3/2} - \frac{t^{5/2}}{5/2} + C$ p373#54

Another solution: let $u = \sqrt{t}$, so $t = u^2$ and $dt = 2u\,du$, giving $\displaystyle\int (2u^2 - 2u^4)\,du = \frac{2u^3}{3} - \frac{2u^5}{5} + C$

(c) $\displaystyle\int_0^1 2xe^{x^2}\,dx = \int_0^1 e^u\,du = e^u\Big|_0^1 = e^1 - e^0 = e - 1$ p412#39

Use the substitution $u = x^2$. Then $du = 2x\,dx$, and we can change the limits too: when $x = 0$ we have $u = 0$, and when $x = 1$ we have $u = 1$.

3. (15 pts) Find the following limits algebraically.

(a) $\displaystyle\lim_{x\to 4} \frac{x^2 - 16}{x^2 - x - 12} = \lim_{x\to 4} \frac{(x-4)(x+4)}{(x-4)(x+3)} = \lim_{x\to 4} \frac{x+4}{x+3} = \frac{8}{7}$ p113#22

(b) $\displaystyle\lim_{x\to -1} \frac{4x^2 + 5x - 7}{3x^2 - 2x + 1} = \lim_{x\to -1} \frac{4(-1)^2 + 5(-1) - 7}{3(-1)^2 - 2(-1) + 1} = \frac{4-5-7}{3+2+1} = \frac{-8}{6} = -\frac{4}{3}$ p113#20

4. (25 pts) Let $f(x) = x + \dfrac{4}{x}$. p229 Example 12

(a) Find $f'(x)$ and $f''(x)$.

(b) Find the critical points and undefined values of $f(x)$.

(c) Determine where $f(x)$ is increasing and/or decreasing; find the relative maximum and minimum values of $f(x)$ (use the second derivative test to check your answers).

(d) Graph the function $f(x)$, using your knowledge of calculus.

The complete solution is given in the text: see Example 12, page 229.

5. (15 pts) A firm estimates that its daily total cost function (in suitable units) is $C(x) = x^3 - 6x^2 + 13x + 15$, and its total revenue function is $R(x) = 28x$. Find the value of x that maximizes the daily profit.

Profit $\quad P(x) = R(x) - C(x) = 28x - (x^3 - 6x^2 + 13x + 15)15x - x^3 + 6x^2 - 15.$

$P'(x) = 15 - 3x^2 + 12x = (-3)(x^2 - 4x - 5) = (-3)(x-5)(x+1)$

Setting $P'(x) = 0$, we get $x = 5$ or $x = -1$. The negative solution is not possible; $P''(x) = -6x + 12$, and so $P''(5) = -18$, showing that $x = 5$ does yield the maximum daily profit.

6. (15 pts) TWA requires that the total dimensions (length + width + height) of a checked bag cannot exceed 62 inches. Suppose that you want to check a bag whose height equals its width. What are the dimensions of the largest bag of this shape that you can check on a TWA flight? (Maximize the volume)

Let x be the width, and let y be the length. Since the height equals the width, for the volume we get the formula $V = x^2y$. The requirement on dimensions shows that the constraint equation is $2x + y = 62$, or $y = 62 - 2x$. This gives the function for the volume: $V(x) = x^2(62 - 2x)$.

$V(x) = 62x^2 - 2x^3$ $\quad V'(x) = 124x - 6x^2$
Set $V'(x) = 0$ $\quad 124x - 6x^2 = 0$ $\quad 2x(62 - 3x) = 0$ $\quad x = \frac{62}{3}$

This value does give a maximum, since $V''(x) = 124 - 12x$, and so $V''(\frac{62}{3})$ is negative.

The length is $y = 62 - 2x = 62 - 2(\frac{62}{3}) = 62(1 - \frac{2}{3}) = \frac{62}{3}$. (The most efficient shape is a cube.)

7. (15 pts) Find the area bounded by the curve $y = \dfrac{1}{x}$ and the lines $y = x$, $x = 1$, and $x = e$.

$$\text{Area} = \int_1^e \left(1 - \frac{1}{x}\right) dx = \frac{x^2}{2} - \ln x \Big|_1^e = \left(\frac{e^2}{2} - \ln e\right) - \left(\frac{1^2}{2} - \ln 1\right) = \frac{e^2}{2} - 1 - \frac{1}{2} = \frac{e^2}{2} - \frac{3}{2}$$

8. (10 pts) A company finds that the rate a which consumer-demand quantity changes with respect to price is given by the marginal-demand function $D'(p) = -\dfrac{4000}{p^2}$. Find the demand function if it is know that 1003 units of the product are demanded by consumers when the price is \$4 per unit. p372#35

Since $D'(p) = -4000p^{-2}$, the antiderivative is $D(p) = -4000 \cdot \dfrac{p^{-1}}{-1} + C = \dfrac{4000}{p} + C$.

When $p = 4$, we get $1003 = D(4) = \dfrac{4000}{4} + C$, so $C = 3$. Answer: $D(p) = \dfrac{4000}{p} + 3$.

9. (10 pts) Find $f'(x)$ and $f''(x)$ for the function $f(x) = x^2 \ln x$. *Use the product rule!* p317#109

$f'(x) = (2x)\ln x + x^2\left(\dfrac{1}{x}\right) = 2x\ln x + x$ $\qquad f''(x) = (2)\ln x + 2x\left(\dfrac{1}{x}\right) + 1 = 2\ln x + 3$

10. (15 pts) The power supply of a satellite is a radioisotope. The power output P, in watts, decreases at a rate proportional to the amount present; P is given by $P(t) = 50e^{-0.004t}$, where t is the time, in days. *You may leave your answers in terms of e^x and/or $\ln x$.* p343#41

(a) How much power is available after 250 days?

$P(250) = 50e^{-0.004\cdot 250} = 50e^{-1}$

(b) What is the half-life of the power supply?

Solving $P(t) = 25$ we get $25 = 50e^{-0.004t}$ $\quad .5 = e^{-0.004t}$ $\quad \ln(.5) = -0.004t$ $\quad t = -\dfrac{\ln(.5)}{.004}$

11. (10 pts) Using the limit definition of the derivative (not the power formula), find the derivative $f'(x)$ of $f(x) = \dfrac{3}{x}$. p131 Example 4

The complete solution is given in the text: see Example 4 on page 131.