Math 211 C Review for EXAM II on 3/21/03

#2,4,6 from Exam II, 10/18/02

2. (15 pts; p200 #23) The graph of $f(x) = \frac{-8}{x^2+1}$ has one relative extreme point. Find the coordinates of this point, and use the sign of $f'(x)$ to determine whether the point is a relative maximum or a relative minimum. *You do not need to include a sketch of the graph.*

4. (20 pts) Sketch the graph of the function $f(x) = x^3 - 3x^2 - 1$. First find:
   (a) $f'(x)$ and $f''(x)$ (factor your answer);
   (b) critical points (if any); where $f(x)$ is increasing; where $f(x)$ is decreasing;
   (c) inflection points (if any); where $f(x)$ is concave up; where $f(x)$ is concave down;
   (d) the extreme points (relative maximum or relative minimum) of the curve (if any).

6. (15 pts; p232 #41) For the function $f(x) = \frac{2x^2}{x^2-16}$, find
   (a) $f'(x) =$
   (b) critical points (if any); where the graph is increasing; where the graph is decreasing;
   (c) the vertical and horizontal asymptotes.
   (d) Given $f''(x) = \frac{192x^2+1024}{(x^2-16)^3}$, find where the graph is concave up; concave down.

#1,2,6 from Exam III, 11/15/02

1. (10 pts; see p 243 Example 6)
Find the absolute maximum and minimum values of $f(x) = 5x + \frac{35}{x}$ on the interval $[1, 5]$.

2. (15 pts; p 254 Example 4)
A stereo manufacturer determines that in order to sell x units of a new stereo, the price per unit must be $p = 1000 - x$. The manufacturer also determines that the total cost of producing x units is given by $C(x) = 3000 + 20x$. How many units must the company produce and sell in order to maximize profit?

6. (20 pts; p 263 #33)
A rectangular box with a volume of 320 ft^3 is to be constructed with a square base and top. The cost per square foot for the bottom is 15 cents, for the top is 10 cents, and for the sides is 2.5 cents. What dimensions will minimize the cost?

2. (15 pts; p200 #23) The graph of $f(x) = \dfrac{-8}{x^2+1}$ has one relative extreme point. Find the coordinates of this point, and use the sign of $f'(x)$ to determine whether the point is a relative maximum or a relative minimum. *You do not need to include a sketch of the graph.*

Using the quotient rule (remember that the derivative of a constant is zero) you should get $f'(x) = \dfrac{(0)(x^2+1)-(-8)(2x)}{(x^2+1)^2} = \dfrac{16x}{(x^2+1)^2}$. Since the denominator is always positive, the sign of $f'(x)$ just depends on the sign of x, so $f'(x)$ is negative when $x < 0$ and positive when $0 < x$. This means that $(0,0)$ is a relative minimum point.

If you decide to use the product rule, $f(x) = -8(x^2+1)^{-1}$, and then $f'(x) = (8)(x^2+1)^{-1}(2x)$, which gives the same answer as we got before.

If you set $f'(x) = 0$, you get $\dfrac{16x}{(x^2+1)^2} = 0$. Then you can solve the equation in two different ways.
You could say that a fraction is zero only when the numerator is zero, so you get $16x = 0$, or $x = 0$.
Or, since $(x^2+1)^2$ is nonzero for all x, you could multiply both sides of the equation by $(x^2+1)^2$.
$\dfrac{16x}{(x^2+1)^2} \cdot (x^2+1)^2 = 0 \cdot (x^2+1)^2$ gives the same answer: $16x = 0$, and then $x = 0$.

4. (20 pts) Sketch the graph (5 pts) of the function $f(x) = x^3 - 3x^2 - 1$. First find:
(a) (4 pts) $f'(x)$ and $f''(x)$ (factor your answer);

$f'(x) = 3x^2 - 6x = 3x(x-2)$ $\qquad$ $f''(x) = 6x - 6 = 6(x-1)$

(b) (4 pts) critical points (if any); where $f(x)$ is increasing; where $f(x)$ is decreasing;

Setting $f'(x) = 0$ we get $x = 0$ or $x = 2$.
We need to check the sign of $f'(x)$ in three intervals: for $x < 0$; for $0 < x < 2$; and for $2 < x$. We can choose the points $x = -1$, $x = 1$, and $x = 3$ in these intervals. Then $f(-1) = 9$, $f'(1) = -3$, and $f'(3) = 9$, so we can conclude that
$f(x)$ is increasing for $x < 0$ $\quad$ $f(x)$ is decreasing for $0 < x < 2$ $\quad$ $f(x)$ is increasing for $2 < x < 0$

(c) (3 pts) inflection points (if any); where $f(x)$ is concave up; where $f(x)$ is concave down;

Setting $f''(x) = 0$ we get $x = 1$, and this gives an inflection point of $(1,-3)$ since $f''(x)$ changes sign. The graph is concave down for $x < 1$, and concave up for $1 < x$.

(d) (4 pts) the extreme points (relative maximum or relative minimum) of the curve (if any).

For the critical points $(0,-1)$ and $(2,-5)$, the curve is concave down at the first one since $f''(0)$ is negative, and concave up at the second since $f''(2)$ is positive. We conclude that $x = 0$ produces a relative maximum, while $x = 2$ gives a relative minimum.

6. (15 pts; p232 #41) For the function $f(x) = \dfrac{2x^2}{x^2-16}$, find

(a) (4 pts) $f'(x) = \dfrac{4x(x^2-16) - 2x^2(2x)}{(x^2-16)^2} = \dfrac{4x^3 - 64x - 4x^3}{(x^2-16)^2} = \dfrac{-64x}{(x^2-16)^2}$

(b) (4 pts) critical points (if any); where the graph is increasing; where the graph is decreasing;

The only way a fraction can be equal to zero is if the numerator is zero.
Setting $f'(x) = 0$ gives $-64x = 0$, so $x = 0$ is the only critical point.
Then it is easy to see that $f'(x)$ is positive for $x < 0$ and negative for $0 < x$.
Conclusion: $f(x)$ is increasing for $x < 0$ and decreasing for $0 < x$.

(c) (4 pts) the vertical and horizontal asymptotes.

Vertical asymptotes: setting the denominator equal to zero gives $x = 4$ and $x = -4$.

Horizontal asymptote: $y = 2$ since $\displaystyle\lim_{x\to\infty} \frac{2x^2}{x^2-16} = \lim_{x\to\infty} \frac{2}{1-16/x^2} = 2$.

(d) (3 pts) Given $f''(x) = \dfrac{192x^2 + 1024}{(x^2-16)^3}$, find where the graph is concave up; concave down.

The numerator is positive for all values of x, so $f''(x)$ can only change sign in its denominator. When $-4 < x < 4$, the denominator is negative, and so $f''(x)$ is negative, and $f(x)$ is concave down in this interval. The graph is concave up for $x < -4$ and for $4 < x$, since on these intervals we have $x^2 - 16 > 0$, and therefore $f''(x)$ is positive.

MATH 211 C EXAM III SOLUTIONS 11/15/2002

1. (10 pts; see p 243 Example 6) Find the absolute max and min values of $f(x) = 5x + \dfrac{35}{x}$ on $[1, 5]$.

$f'(x) = 5 - 35x^{-2} = 5 - \dfrac{35}{x^2}$ and setting $f'(x) = 0$ gives $x = \pm\sqrt{7}$. The only critical point in $[1, 5]$ is $x = \sqrt{7}$. We have $f(1) = 40$, $f(5) = 32$, and $f(\sqrt{7}) = 5\sqrt{7} + \dfrac{35}{\sqrt{7}} = 5\sqrt{7} + \dfrac{35\sqrt{7}}{\sqrt{7}\sqrt{7}} = 5\sqrt{7} + 5\sqrt{7} = 10\sqrt{7} < 10 \cdot 3 = 30$, so $f(\sqrt{7}) = 10\sqrt{7}$ is the absolute minimum on $[1, 5]$, while $f(1) = 40$ is the absolute maximum.

2. (15 pts; p 254 Example 4) A stereo manufacturer determines that in order to sell x units of a new stereo, the price per unit must be $p = 1000 - x$. The total cost of producing x units is given by $C(x) = 3000 + 20x$. How many units must the company produce and sell in order to maximize profit?

$P(x) = R(x) - C(x) = x(1000 - x) - (3000 - 20x) = -x^2 + 980x - 3000$. Setting $P'(x) = 0$ gives the critical point $x = 490$, and this *does* produce a maximum since $P''(x) = -2$ means that the graph is concave down for all x.

6. (20 pts; p 263 #33) A rectangular box with a volume of 320 ft^3 is to be constructed with a square base and top. The cost per square foot for the bottom is 15 cents, for the top is 10 cents, and for the sides is 2.5 cents. What dimensions will minimize the cost?

Total cost: $C(x) = 0.15x^2 + 0.1x^2 + 0.1xy = 0.25x^2 + 0.1xy$.

We have $x^2y = 320$, so $C(x) = 0.25x^2 + 0.1xy = 0.25x^2 + 0.1x\left(\dfrac{320}{x^2}\right) = 0.25x^2 + \dfrac{32}{x}$.

Setting $C'(x) = .5x - \dfrac{32}{x^2} = 0$, we get $x^3 = 64$, so $x = 4$.

This does minimize cost since $C''(x) = .5 + \dfrac{64}{x^3}$ is positive for $x = 4$.

Answer: $x = 4$ and $y = \dfrac{320}{(4)^2} = \dfrac{320}{16} = 20$.