MATH 211 C | EXAM III | Solutions
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1. (6 pts each) Find the derivative of each of these functions.

(a) $f(x) = e^{-x} + xe^{-2x} + x^3$ Use the formula $\frac{d}{dx}e^{u(x)} = e^{u(x)} \cdot u'(x)$. (4.1 #29)

$f'(x) = e^{-x} \cdot (-1) + \left[1 \cdot e^{-2x} \ + \ x \cdot (e^{-2x} \cdot (-2))\right] + 3x^2$ Use the product rule on the middle term.

(b) $f(x) = \ln(5x^2 - 7)$ Use the formula $\frac{d}{dx}\ln(u(x)) = \frac{u'(x)}{u(x)}$. (4.2 #51)

$$f'(x) = \frac{10x}{5x^2 - 7}$$

(c) $f(x) = \ln\left(x + \sqrt{1 + x^2}\right) = \ln\left(x + (1 + x^2)^{1/2}\right)$ Note: you can't simplify $\ln(M + N)$. (4.2 #89)

$$f'(x) = \frac{1 + (1/2)(1 + x^2)^{-1/2}(2x)}{x + (1 + x^2)^{1/2}}$$

(d) $f(x) = e^{\sqrt{x}} + \sqrt{e^x}$ (*algebra*) $= e^{x^{1/2}} + (e^x)^{1/2} = e^{x^{1/2}} + e^{(1/2)x}$ (4.1 #63)

$$f'(x) = e^{x^{1/2}} \cdot \frac{d}{dx}(x^{-1/2}) \ + \ e^{(1/2)x} \cdot \frac{d}{dx}(1/2)x = e^{x^{1/2}} \cdot (1/2)x^{-1/2} \ + \ e^{(1/2)x} \cdot (1/2)$$

(e) $f(x) = \ln\left[\frac{x^5}{(x + 5)^2}\right]$ (*algebra*) $= \ln(x^5) \ - \ \ln(x + 5)^2 = 5\ln x \ - \ 2\ln(x + 5)$ (4.2 #83)

$$f'(x) = 5 \cdot \frac{1}{x} - 2 \cdot \frac{1}{x + 5} = \frac{5}{x} - \frac{2}{x + 5}$$

2. (20 pts) Find the following antiderivatives and integrals.

(a) (7 pts) $\int \left(x^2 - \frac{3}{2}\sqrt{x} + x^{-4/3}\right) \, dx = \int \left(x^2 - \frac{3}{2}x^{1/2} + x^{-4/3}\right) \, dx$ (5.1 #25)

$$= \frac{x^3}{3} - \frac{3}{2} \cdot \frac{x^{3/2}}{3/2} + \frac{x^{-1/3}}{-1/3} + C = \frac{x^3}{3} - \frac{3}{2} \cdot \frac{3}{2} \cdot x^{3/2} + \frac{-3}{1} \cdot x^{-1/3} + C = \frac{x^3}{3} - x^{3/2} - 3x^{-1/3} + C$$

(b) (6 pts) Find $C(x)$, given that $C'(x) = x^3 - 2x$ and the fixed cost is $C(0) = 100$. (5.1 #31)

Find the antiderivative: $C(x) = \frac{x^4}{4} - x^2 + K$.

Substituting $x = 0$, we get $100 = C(0) = K$, so the answer is $C(x) = \frac{x^4}{4} - x^2 + 100$.

(c) $\int_1^e \left(x + \frac{1}{x}\right) \, dx = \frac{x^2}{2} + \ln x\Big|_1^e = \left(\frac{e^2}{2} + \ln e\right) - \left(\frac{1^2}{2} + \ln 1\right) = \frac{e^2}{2} + 1 - \frac{1}{2} - 0 = \frac{e^2}{2} + \frac{1}{2}$ (5.2 #41)

3. (10 pts; 5.3 #35) The temperature over a 10 hour period is given by $f(t) = -t^2 + 5t + 40$, for $0 \le t \le 10$. Find the average temperature during this period.

$$\text{Avg temp} = \frac{1}{10 - 0}\int_0^{10} f(t)\, dt = \frac{1}{10}\int_0^{10} (-t^2 + 5t + 40)\, dt = \frac{1}{10}\left(-\frac{t^3}{3} + 5 \cdot \frac{t^2}{2} + 40t\right)\Bigg|_0^{10}$$

$$= \frac{1}{10}\left(-\frac{(10)^3}{3} + \frac{5(10)^2}{2} + 40(10) \ - \ 0\right) = -\frac{(10)^2}{3} + \frac{5(10)}{2} + 40 = -\frac{100}{3} + 25 + 40 = -33\tfrac{1}{3} + 65 = 31\tfrac{2}{3}$$

4. (20 pts) (a) Find the area under the curve $y = x^2 + 1$, over the interval $[1, 3]$.

$$\int_1^3 (x^2+1)\,dx = \left.\frac{x^3}{3} + x\right|_1^3 = \left(\frac{3^3}{3}+3\right) - \left(\frac{1^3}{3}+1\right) = \left(\frac{27}{3}+3\right) - \left(\frac{1}{3}+1\right) = 12 - 1\frac{1}{3} = 10\frac{2}{3}$$

(b) Approximate the area under the curve $y = x^2 + 1$ over the interval $[1, 3]$ by computing the area of each rectangle in the given graph and then adding.

The four rectangles in the graph are found from the heights of the curve at 1, 1.5, 2, and 2.5. The heights are $f(1) = 1^2 + 1 = 2$, $f(1.5) = (1.5)^2 + 1 = 2.25 + 1 = 3.25$, $f(2) = (2)^2 + 1 = 5$, $f(2.5) = (2.5)^2 + 1 = 6.25 + 1 = 7.25$. Multiply each height by the width .5 and add, to get the total area of 8.75. This is an approximation to the exact area of $10.66\ldots$ found in part (a).

5. (10 pts) A colony of sea monsters is growing at a rate proportional to its size. Observations show that the population is tripling every 11 years. How many years will it take for the population to become 6 times its original size?

In the equation $P(t) = P_0 e^{kt}$ for exponential growth, we are given that $P(11) = 3P_0$, and we must solve the equation $P(t) = 6P_0$.

$$P_0 e^{k(11)} = 3P_0 \qquad e^{k(11)} = 3 \qquad \ln(e^{k(11)}) = \ln 3 \qquad 11k = \ln 3 \qquad k = \frac{1.10}{11} = .1$$

$$P_0 e^{.1t} = 6P_0 \qquad e^{.1t} = 6 \qquad \ln(e^{.1t}) = \ln 6 \qquad .1t = \ln 6 \qquad t = \frac{1.79}{.1} = 17.9 \text{ years}$$

Note: Since the population triples every 11 *years, in* 22 *years the population will be* $9P_0$. *If the growth curve was just a straight line, the population would get to* $6P_0$ *halfway between* 11 *and* 22 *years, but since the growth is exponential, the population grows faster at the end of that time period than it does at the beginning. That is why the answer is* 17.9 *years instead of* 16.5 *years.*

6. (10 pts; 4.4 #35) Newton's law of cooling leads to the formula $T(t) = ae^{-kt} + C$ for the temperature at time t, where C is the constant temperature of the surrounding medium.
Suppose that the temperature of a hot liquid is 100° and the room temperature is 75°. The liquid cools to 90° in 10 minutes.

(a) Find the value of the constant a. (b) Find the value of the constant k.

The problem gives us the information that $C = 75$, $T(0) = 100$, and $T(10) = 90$. To find a, set $t = 0$. Then to find k, set $t = 10$.

$$100 = T(0) = ae^{-k(0)} + 75 \qquad 100 = a \cdot (1) + 75 \qquad a = 100 - 75 = 25$$

$$90 = T(10) = 25e^{-k(10)} + 75 \qquad 15 = 25e^{-k(10)} \qquad e^{-k(10)} = \frac{15}{25} = \frac{3}{5} \qquad \ln(e^{-k(10)}) = \ln\frac{3}{5}$$

$$-10k = \ln\frac{3}{5} \qquad -10k = \ln 3 - \ln 5 \qquad -10k = 1.10 - 1.61 \qquad k = \frac{-.51}{-10} = .051$$