1. (30 points) Find the derivative of each of these functions. You do not need to simplify your answers.

(a) (p108 #29) $f(x) = \frac{x}{3} + \frac{3}{x} = \frac{1}{3}x + 3x^{-1}$ $f'(x) = \frac{1}{3} - 3x^{-2}$

(b) (p215 #15) $f(x) = 2(x^3 - 1)(3x^2 + 1)^4$ $f'(x) = 2(3x^2)(3x^2 + 1)^4 + 2(x^3 - 1)(4)(3x^2 + 1)^3(6x)$

(c) $f(x) = \ln(x^4 - e^x)$ $f'(x) = \frac{4x^3 - e^x}{x^4 - e^x}$

(d) $f(x) = \frac{e^{x^2-x}}{e^x + 1}$ $f'(x) = \frac{(e^{x^2-x})(2x-1)(e^x+1) - (e^{x^2-x})(e^x)}{(e^x+1)^2}$

2. (a) (7 pts; p96 #23) $\lim_{x\to 6} \frac{x^2 - 6x}{x^2 - 5x - 6} = \lim_{x\to 6} \frac{(x-6)(x)}{(x-6)(x+1)} = \lim_{x\to 6} \frac{x}{x+1} = \frac{6}{7}$

(b) (8 pts) $\lim_{x\to 5} \frac{x^2 - 10x + 25}{x^2 - 25} = \lim_{x\to 5} \frac{(x-5)(x-5)}{(x-5)(x+5)} = \lim_{x\to 5} \frac{x-5}{x+5} = \frac{0}{10} = 0$

3. (20 pts; p156, Ex 2) Using the techniques of calculus, sketch the graph of the function $f(x) = x^3 - 3x^2 + 5$. On the graph, indicate all relative extreme points (relative maximum and relative minimum points) and all points of inflection.

$f'(x) = 3x^2 - 6x$ $f''(x) = 6x - 6$ Setting $f'(x) = 0$ gives the potential extreme points $x = 0$, $x = 2$. Since $f''(0) = -6$ and $f''(2) = 6$, there is a local maximum at $(0, 5)$ and a local minimum at $(2, 1)$. Setting $f''(x) = 0$ shows that $(1, 3)$ is a point of inflection. You should also plot $(-1, 1)$ and $(3, 5)$ on the graph.

4. (5 pts) Let $P(t) = 500 - 100e^{-5t}$. When $t = 2$, is $P(t)$ increasing or decreasing? Explain your answer.

We have $P'(t) = -100e^{-5t}(-5)$, so $P'(2) = \frac{500}{e^{10}} > 0$, and so $P(t)$ is increasing when $t = 2$.

5. (10 pts) Find an equation for the line tangent to the curve $y = 2x(x - 4)^6$ at $x = 5$.

When $x = 5$, $y = 2{\cdot}5(1)^6 = 10$. $y' = 2(x-4)^6 + (2x)(6)(x-4)^5$ When $x = 5$, $y' = 2(1)^6 + 2{\cdot}5{\cdot}6(1)^5 = 62$. The equation of the tangent line is $y = 62(x - 5) + 10$.

6. (15 pts; p201 #56) A closed rectangular box with a square base is to be constructed using two different types of wood. The top is made of wood costing \$3 per square foot, and the remainder is made of wood costing \$1 per square foot. Suppose that \$48 is available to spend. Find the dimensions of the box of the greatest volume that can be constructed.

Solution: If the box has a base that is x feet by x feet, with height h, then the problem is to maximize the volume $V = x^2h$ (this is the objective equation). The cost of the box is $3x^2 + x^2 + 4xh = 48$, representing the cost of the top, the bottom, and the four sides, respectively. This gives the constraint equation, which we can solve for h in terms of x.

Objective: $V = x^2h$ Constraint: $3x^2 + x^2 + 4xh = 48$ $x^2 + xh = 12$ $h = \frac{12 - x^2}{x}$

$V = x^2h = x^2\left(\frac{12 - x^2}{x}\right) = x(12 - x^2) = 12x - x^3.$ $V'(x) = 12 - 3x^2$ $V''(x) = -6x$

Setting $V'(x) = 0$ we get $12 - 3x^2 = 0$, or $x^2 = 4$, so $x = \pm 2$.

We have $V''(2) = -12$, so $V(x)$ is concave down at $x = 2$, showing that $x = 2$ does lead to a maximum value for the area. Substituting $x = 2$ in the constraint equation $h = \frac{12 - x^2}{x}$ gives $h = 4$.

7. (12 pts) Ten grams of a certain radioactive material decays to three grams in five years. What is the half-life of the radioactive material?

We use the equation $P(t) = P_0e^{-\lambda t}$, with $P_0 = 10$, the initial amount.

$P(5) = 3$ $10e^{-\lambda \cdot 5} = 3$ $e^{-\lambda \cdot 5} = .3$ $-5\lambda = \ln(.3)$ $\lambda = -\frac{1}{5}\ln(.3) = -.2\ln(.3)$

To find the half-life, we need to solve for t in the eqation $P(t) = 5$.

$P(t) = 5 \qquad 10e^{-\lambda t} = 5 \qquad e^{-\lambda t} = .5 \qquad -\lambda t = \ln(.5) \qquad$ Answer: $\quad t = \dfrac{\ln(.5)}{-\lambda} = \dfrac{\ln(.5)}{.2\ln(.3)} = \dfrac{5\ln(.5)}{\ln(.3)}$

8. (18 pts) The demand equation for a certain product is $p = 180 - 3x$, where p is the price and x is the number of units produced. The cost function is $C(x) = 60 + 80x - x^2$, where $0 \le x \le 40$.

(a) Determine the level of production that will maximize the profit, and determine the corresponding price.

Profit = Revenue - Cost = (price per unit)(# units) - Cost
$P(x) = (180 - 3x)(x) - (60 + 80x - x^2) = 180x - 3x^2 - 60 - 80x + x^2 = -2x^2 + 100x - 60$
$P'(x) = -4x + 100$
Setting $P'(x) = 0$ gives $4x = 100$ or $x = 25$. Since $P''(x) = -4$, this gives a maximum. The corresponding price is $p = 180 - 3(25) = 105$.

(b) Suppose that the government imposes a tax of \$4 per unit produced, increasing the cost by \$4 per unit. Find the new price that now maximizes the profit.

Since the cost is increased by \$4 per unit, we must add $4x$ to the cost function.
$P(x) = (180 - 3x)(x) - (60 + 84x - x^2) = -2x^2 + 96x - 60 \qquad P'(x) = -4x + 96$
Setting $P'(x) = 0$ gives $4x = 96$ or $x = 24$. The corresponding price is $p = 180 - 3(24) = 108$.

9. (30 pts) Find the following integrals.

(a) (p330 #16) $\displaystyle \int_1^8 2x^{1/3}\,dx = \frac{2x^{4/3}}{4/3}\Big|_1^8 = \frac{3}{2}(\sqrt[3]{x})^4\Big|_1^8 = \frac{3\cdot 2^4}{2} - \frac{3\cdot 1^4}{2} = 24 - \frac{3}{2} = 22\frac{1}{2}$

(b) (p330 #27) $\displaystyle \int_2^4 \left(\frac{2}{x^2} - \frac{1}{x+5}\right) dx = \int_2^4 \left(2x^{-2} - \frac{1}{x+5}\right) dx = \frac{2x^{-1}}{-1} - \ln(x+5)\Big|_2^4 =$
$\displaystyle \frac{-2}{x} - \ln(x+5)\Big|_2^4 = \left[\frac{-2}{4} - \ln(4+5)\right] - \left[\frac{-2}{2} - \ln(2+5)\right] = \frac{1}{2} + \ln 7 - \ln 9$

(c) (p449 #8) $\displaystyle \int \frac{(\ln x)^3}{x}\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{1}{4}(\ln x)^4 + C$ (Substitute $u = \ln x$, and $du = \frac{1}{x}dx$.)

10. (15 pts; p335 Example 2) Find the area bounded by the curves $y = x^2 - 4x + 4$ and $y = x^2$ (from $x = 0$ to $x = 3$). First graph the two functions.

See the textbook for the graphs and the solution: $\displaystyle \int_0^1 (-4x+4)\,dx + \int_1^3 (4x-4)\,dx = 2 + 8 = 10$

11. (20 pts) Find the derivative of each of the following functions.

(a) $\quad f(x) = \sqrt{xe^x + x^e} = (xe^x + x^e)^{1/2} \qquad f'(x) = \frac{1}{2}(xe^x + x^e)^{-1/2}(e^x + xe^x + ex^{e-1})$

(b) (p258 #26) $\quad f(x) = \ln\left[\dfrac{e^{4x}\sqrt{3x+1}}{1-x^2}\right] = 4x + \frac{1}{2}\ln(3x+1) - \ln(1-x^2) \qquad f'(x) = 4 + \dfrac{3}{2(3x+1)} + \dfrac{2x}{1-x^2}$

12. (10 pts; p97 #33) Using the limit definition of the derivative, find $f'(x)$ for $f(x) = \dfrac{1}{2x+5}$.

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{1}{2(x+h)+5} - \dfrac{1}{2x+5}}{h} = \lim_{h\to 0}\frac{1}{h}\left[\frac{2x+5}{(2x+2h+5)(2x+5)} - \frac{2x+2h+5}{(2x+2h+5)(2x+5)}\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\cdot\frac{2x+5-2x-2h-5}{(2x+2h+5)(2x+5)} = \lim_{h\to 0}\frac{1}{h}\cdot\frac{-2h}{(2x+2h+5)(2x+5)} = \lim_{h\to 0}\frac{-2}{(2x+2h+5)(2x+5)} = \frac{-2}{(2x+5)^2}$$