1. (30 pts) Find the derivative of each of these functions. You do not need to simplify your answers.

(a) \( p108 \#29 \) \( f(x) = \frac{x}{3} + \frac{3}{x} = \frac{1}{3}x + 3x^{-1} \) \( f'(x) = 3 - 3x^{-2} \)

(b) \( p215 \#15 \) \( f(x) = 2(x^3 - 1)(3x^2 + 1) \) \( f'(x) = 2(3x^2)(3x^2 + 1)^4 + 2(x^3 - 1)(4x^2 + 1)^3(6x) \)

(c) \( f(x) = \ln(x^4 - e^x) \) \( f'(x) = \frac{4x^3 - e^x}{x^4 - e^x} \)

(d) \( f(x) = \frac{e^{x^2-x}}{e^x + 1} \) \( f'(x) = \frac{(e^{x^2-x})(2x-1)(e^x + 1) - (e^{x^2-x})(e^x)}{(e^x + 1)^2} \)

2. (a) \( p96 \#23 \) \( \lim_{x \to 3} \frac{x^2 - 6x}{x^2 - 5x - 6} = \lim_{x \to 3} \frac{(x-6)(x)}{(x-6)(x+1)} = \lim_{x \to 3} \frac{x}{x+1} = \frac{6}{7} \)

(b) \( p215 \#15 \) \( \lim_{x \to 5} \frac{x^2 - 10x + 25}{x^2 - 25} = \lim_{x \to 5} \frac{(x-5)(x-5)}{(x-5)(x+5)} = \lim_{x \to 5} \frac{x-5}{x+5} = \frac{0}{10} = 0 \)

3. (20 pts; p156, Ex 2) Using the techniques of calculus, sketch the graph of the function \( f(x) = x^3 - 3x^2 + 5 \). On the graph, indicate all relative extreme points (relative maximum and relative minimum points) and all points of inflection.

\[ f'(x) = 3x^2 - 6x \]

\[ f''(x) = 6x - 6 \]

Setting \( f'(x) = 0 \) gives the potential extreme points \( x = 0 \), \( x = 2 \). Since \( f''(0) = -6 \) and \( f''(2) = 6 \), there is a local maximum at \((0, 5)\) and a local minimum at \((2, 1)\). Setting \( f''(x) = 0 \) shows that \((1, 3)\) is a point of inflection. You should also plot \((-1, 1)\) and \((3, 5)\) on the graph.

4. (5 pts) Let \( P(t) = 500 - 100e^{-5t} \). When \( t = 2 \), is \( P(t) \) increasing or decreasing? Explain your answer.

We have \( P'(t) = -100e^{-5t} \), so \( P'(2) = \frac{500}{e^{10}} > 0 \), and so \( P(t) \) is increasing when \( t = 2 \).

5. (10 pts) Find an equation for the line tangent to the curve \( y = 2x(x - 4)^6 \) at \( x = 5 \).

When \( x = 5 \), \( y = 2(5 - 4)^6 = 10 \). \( y' = 2(x - 4)^6 + 2(2x)(6)(x - 4)^5 \) When \( x = 5 \), \( y' = 2(1)^6 + 2(5)(1)^5 = 62 \).

The equation of the tangent line is \( y = 62(x - 5) + 10 \).

6. (15 pts; p201 #56) A closed rectangular box with a square base is to be constructed using two different types of wood. The top is made of wood costing \$3 per square foot, and the remainder is made of wood costing \$1 per square foot. Suppose that \$48 is available to spend. Find the dimensions of the box of the greatest volume that can be constructed.

Solution: If the box has a base that is \( x \) feet by \( x \) feet, with height \( h \), then the problem is to maximize the volume \( V = x^2h \) (this is the objective equation). The cost of the box is \( 3x^2 + x^2 + 4xh = 48 \), representing the cost of the top, the bottom, and the four sides, respectively. This gives the constraint equation, which we can solve for \( h \) in terms of \( x \).

Objective: \( V = x^2h \)  
Constraint: \( 3x^2 + x^2 + 4xh = 48 \) \( x^2 + xh = 12 \) \( h = \frac{12 - x^2}{x} \)

\( V = x^2h = x^2 \left( \frac{12 - x^2}{x} \right) = x(12 - x^2) = 12x - x^3 \). \( V'(x) = 12 - 3x^2 \) \( V''(x) = -6x \)

Setting \( V'(x) = 0 \) we get \( 12 - 3x^2 = 0 \), or \( x^2 = 4 \), so \( x = \pm 2 \).

We have \( V''(2) = -12 \), so \( V'(x) \) is concave down at \( x = 2 \), showing that \( x = 2 \) does lead to a maximum value for the area. Substituting \( x = 2 \) in the constraint equation \( h = \frac{12 - x^2}{x} \) gives \( h = 4 \).

7. (12 pts) Ten grams of a certain radioactive material decays to three grams in five years. What is the half-life of the radioactive material?

We use the equation \( P(t) = P_0e^{-\lambda t} \), with \( P_0 = 10 \), the initial amount.

\( P(5) = 3 \) \( 10e^{-\lambda \cdot 5} = 3 \) \( e^{-\lambda \cdot 5} = .3 \) \( -\lambda \cdot 5 = \ln(.3) \) \( \lambda = -\frac{1}{5} \ln(.3) = -.2\ln(.3) \)
To find the half-life, we need to solve for $t$ in the equation $P(t) = \frac{5}{10}e^{-\lambda t} = \frac{5}{10}$. 

$$P(t) = 5 \quad 10e^{-\lambda t} = 5 \quad e^{-\lambda t} = \frac{1}{2} \quad -\lambda t = \ln(0.5) \quad \text{Answer: } t = \frac{\ln(0.5)}{-\lambda} = \frac{\ln(0.5)}{2\ln(3)} = \frac{5\ln(0.5)}{\ln(3)}$$

8. (18 pts) The demand equation for a certain product is $p = 180 - 3x$, where $p$ is the price and $x$ is the number of units produced. The cost function is $C(x) = 60 + 80x - x^2$, where $0 \leq x \leq 40$.

(a) Determine the level of production that will maximize the profit, and determine the corresponding price.

$$\text{Profit} = \text{Revenue} - \text{Cost} = (\text{price per unit})(\text{# units}) - \text{Cost}$$

$$P(x) = (180 - 3x)(x) - (60 + 80x - x^2) = 180x - 3x^2 - 60 - 80x + x^2 = -2x^2 + 100x - 60$$

Setting $P'(x) = 0$ gives $x = 100$ or $x = 4$. Since $P''(x) = -4$, this gives a maximum. The corresponding price is $p = 180 - 3(25) = 105$.

(b) Suppose that the government imposes a tax of $4 per unit produced, increasing the cost by $4 per unit.

Since the cost is increased by $4 per unit, we must add $4x$ to the cost function.

$$P(x) = (180 - 3x)(x) - (60 + 84x - x^2) = -2x^2 + 96x - 60 \quad P'(x) = -4x + 96$$

Setting $P'(x) = 0$ gives $4x = 96$ or $x = 24$. The corresponding price is $p = 180 - 3(24) = 108$.

9. (30 pts) Find the following integrals.

(a) $\int_{1}^{8} \frac{2x^2/3}{4/3} \, dx = \frac{2x^4/3}{4/3} \bigg|_{1}^{8} = \frac{2}{3} (\sqrt[3]{x})^4 \bigg|_{1}^{8} = \frac{3 \cdot 2^4}{2} - \frac{3 \cdot 1^4}{2} = 24 - \frac{3}{2} = 22.5$  

(b) $\int_{2}^{4} \left( \frac{2}{x^2} - \frac{1}{x+5} \right) \, dx = \int_{2}^{4} \left( 2x^{-2} - \frac{1}{x+5} \right) \, dx = \frac{2x^{-1}}{-1} - \ln(x+5) \bigg|_{2}^{4} = -2 - \ln(5) - \left[ -\frac{2}{4} - \ln(2) \right] = -\frac{5}{2} + \ln(7) - \ln(9)$

(c) $\int_{x}^{(\ln x)^3} x \, dx = \int u^3 \, du = \frac{u^4}{4} + C = \frac{1}{4} (\ln x)^4 + C$ (Substitute $u = \ln x$, and $du = \frac{1}{x} \, dx$.)

10. (15 pts; p335 Example 2) Find the area bounded by the curves $y = x^2 - 4x + 4$ and $y = x^2$ (from $x = 0$ to $x = 3$). First graph the two functions.

See the textbook for the graphs and the solution: $\int_{0}^{1} (-4x + 4) \, dx + \int_{1}^{3} (4x - 4) \, dx = 2 + 8 = 10$

11. (20 pts) Find the derivative of each of the following functions.

(a) $f(x) = \sqrt{x^2 + x^5} = (x^2 + x^5)^{1/2}$, $f'(x) = \frac{1}{2} (x^2 + x^5)^{-1/2} (2x + 5x^4 + 5x^4) - 1/2$  

(b) $f(x) = \ln \left( \frac{e^{4x} \sqrt{3x + 1}}{1 - x^2} \right) = 4x + \frac{1}{2} \ln(3x + 1) - \ln(1 - x^2)$  

$f'(x) = 4 + \frac{3}{2(3x + 1)} + \frac{2x}{1 - x^2}$

12. (10 pts; p97 #33) Using the limit definition of the derivative, find $f'(x)$ for $f(x) = \frac{1}{2x + 5}$.

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left[ \frac{2x + 5}{2(x + h) + 5} - \frac{2x + 5}{2(x + h) + 5} \right] = \lim_{h \to 0} \frac{1}{h} \left( \frac{2x + 5}{2(x + h) + 5} - \frac{2x + 5}{2(x + h) + 5} \right) = \lim_{h \to 0} \frac{1}{h} \left( \frac{2x + 5}{2(x + h) + 5} - \frac{2x + 5}{2(x + h) + 5} \right) = \frac{-2}{(2x + 5)^2}$$