1. (30 points) Find the derivative of each of these functions.

   (a) \( f(x) = (x^3 + 8x - 2)^{10} \) Use the general power rule on page 103: \( f'(x) = 10(x^3 + 8x - 2)^9(3x^2 + 8) \)

   (b) \( f(x) = (x + 1)^2(x - 3)^3 \) Use the product rule: \( f'(x) = 2(x + 1)(x - 3)^3 + (x + 1)^2(3)(x - 3)^2 \)

   Note: I recommend this version of the product rule: \( \frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \)

   This pattern may be even clearer for three functions: \( \frac{d}{dx}(uvw) = u'vw + uv'w + uvw' \)

   (c) \( f(x) = \frac{4x^2 + x}{\sqrt{x}} = \frac{4x^2}{x^{1/2}} + \frac{x}{x^{1/2}} = 4x^{3/2} + x^{1/2} \) \( f'(x) = 4(3/2)x^{1/2} + (1/2)x^{-1/2} \)

   (d) \( f(x) = \frac{x^2 - 1}{x^2 + 1} = (x^2 - 1) \cdot (x^2 + 1)^{-1} \) \( f'(x) = (2x) \cdot (x^2 + 1)^{-1} + (x^2 - 1) \cdot (-1)(x^2 + 1)^{-2}(2x) \)

   (e) \( f(x) = \sqrt{\frac{x^2 - 1}{x^2 + 1}} = (x^2 - 1)^{1/2} \cdot (x^2 + 1)^{-1/2} \)

   \( f'(x) = (1/2)(x^2 - 1)^{-1/2}(2x) \cdot (x^2 + 1)^{-1/2} + (x^2 - 1)^{1/2} \cdot (-1/2)(x^2 + 1)^{-3/2}(2x) \)

2. (10 points) The graph of \( f(x) = \frac{-1}{x^2 + 1} \) has one relative extreme point. Find the coordinates of this point, and use the sign of \( f'(x) \) to determine whether the point is a relative maximum or a relative minimum.

   This problem is similar to #44, page 215, and to part of #24, page 215.

   \( f(x) = (-1)(x^2 + 1)^{-1} \) \( f'(x) = (-1)(-1)(x^2 + 1)^{-2}(2x) = \frac{2x}{(x^2 + 1)^2} \)

   Setting \( f'(x) = 0 \) gives the equation \( \frac{2x}{(x^2 + 1)^2} = 0 \), which has the solution \( x = 0 \). When \( x = 0 \), the corresponding point on the curve is \( y = f(0) = \frac{-1}{0^2 + 1} = -1 \), and so the next step is to decide if the critical point \((0, -1)\) represents a relative maximum or a relative minimum.

   We have two ways to answer this question: we can either use the sign of the first derivative to tell where the function is increasing or decreasing, or we can use the sign of the second derivative to tell where the curve is concave up or concave down. In this case it looks like taking the second derivative will be more complicated than using the first derivative.

   Since the denominator of \( f'(x) = \frac{2x}{(x^2 + 1)^2} \) is always positive, the sign depends on the sign of the numerator.

   If \( x < 0 \), then \( f'(x) \) is positive, and \( f(x) \) is increasing.

   If \( x > 0 \), then \( f'(x) \) is negative, and \( f(x) \) is decreasing.

   This says that \( f(x) \) has a relative minimum value at \( x = 0 \).

3. (15 points) The demand equation for a monopolist is \( p = 200 - 3x \) dollars, where \( p \) is the price per unit and \( x \) is the number of units produced. The cost function is \( C(x) = 75 + 80x - x^2 \) dollars for \( 0 \leq x \leq 40 \). Determine the value of \( x \) and the corresponding price per unit that will maximize the total profit.

   This was an assigned homework problem: #14, page 196.

   Profit = Revenue - Cost = price \cdot number - Cost

   \( P(x) = R(x) - C(x) = p \cdot x - C(x) = (200 - 3x)x - (75 + 80x - x^2) = 200x - 3x^2 - 75 - 80x + x^2 \)

   \( P(x) = -2x^2 + 120x - 75 \) \( P'(x) = -4x + 120 \)

   Now set \( P'(x) = 0 \), so \(-4x + 120 = 0 \), and the solution is \( x = 30 \). The corresponding price is \( p = 200 - 3(30) = 110 \).

   To complete the problem you need to show the work that guarantees that you have answered the question: determine the value of \( x \) that will maximize the profit. You can do this either by stating that the graph of \( P(x) \) is a parabola that opens down, or by computing \( P''(0) = -4 \), which shows that the graph is concave down at \( x = 0 \).
4. (20 points) Sketch the graph of \( y = \frac{1}{x} + \frac{1}{4}x + 1 \). Find the intervals on which the graph is concave up or concave down. On the graph, indicate all relative extreme points.

This is the curve in #23, page 168, shifted up 1.

To save space I will give only a partial answer, not including the graph. You should note that the \( y \)-axis is a vertical asymptote, while the line \( y = \frac{1}{4}x + 1 \) is an oblique asymptote.

\[
y = \frac{1}{x} + \frac{1}{4}x + 1 = x^{-1} + \frac{1}{4}x + 1
\]

\[
y' = (-1)x^{-2} + \frac{1}{4} = -\frac{1}{x^2} + \frac{1}{4}
\]

\[
y'' = (-1)(-2)x^{-3} = \frac{2}{x^3}
\]

To find the critical points, set \( y' = 0 \).

\[
-\frac{1}{x^2} + \frac{1}{4} = 0 \quad \frac{1}{x^2} = \frac{1}{4} \quad x^2 = 4 \quad x = \pm 2
\]

When \( x = 2 \), \( y'' \) is positive, so \( y \) is concave up at \((2, 2)\), and \((2, 2)\) is a relative minimum.

When \( x = -2 \), \( y'' \) is negative, so \( y \) is concave down at \((-2, 0)\), and \((-2, 0)\) is a relative minimum.

Intervals on which \( y \) is concave up and concave down: this depends on the sign of \( y'' = \frac{2}{x^3} \), which is negative when \( x \) is negative and positive when \( x \) is positive. There is no point of inflection, since \( y'' = 0 \) has no solution.

- When \( x < 0 \), we have \( y'' < 0 \), so \( y \) is concave down whenever \( x < 0 \).
- When \( x > 0 \), we have \( y'' > 0 \), so \( y \) is concave up whenever \( x < 0 \).

5. (10 points) Find the equation of the line tangent to the graph of \( y = x(x - 1)^5 \) at the point \((2, 2)\).

This is similar to #32, page 209.

We use the product rule and general power rule to get \( y' = (x - 1)^5 + x(5)(x - 1)^4 \). Substituting \( x = 2 \) gives \((2 - 1)^5 + 2 \cdot 5 \cdot (x - 1)^4 = 11\), the slope of the curve at \((2, 2)\).

The equation of the tangent line at \((2, 2)\) is \( y = 11(x - 2) + 2 \).

6. (15 points) A farmer has $1500 available to build an E-shaped fence along a straight river to make two rectangular pastures. The materials for the side parallel to the river cost $6 per foot and the materials for the three sections perpendicular to the river cost $5 per foot. Find the dimensions that maximize the total area.

This was an assigned homework problem: #13, page 176.

The objective is to maximize the area.

Let \( x \) be the length of one of the sides perpendicular to the river, and let \( y \) be the length of the side parallel to the river.

Objective: maximize \( A = xy \)

The additional information about the cost determines the constraint equation: \( 3(5x) + 6y = 1500 \). Solving for \( y \) and substituting into \( A \) gives the next equation.

\[
A = x(250 - (5/2)x) = 250x - (5/2)x^2.
\]

\[
A'(x) = 250 - 5x \quad \text{Set } 250 - 5x = 0
\]

\[
x = 50 \quad \text{The corresponding value for } y \text{ is } y = 250 - (5/2)(50) = 125.
\]

To complete the solution, you must be certain that you have found a maximum. The easiest way to check is to note that the graph of \( A(x) \) is a parabola opening down. You could also use \( A''(x) \) to check.

There were some high scores on the exam, but I feel that most of the class could benefit from another chance to work maximum and minimum problems.

On Friday, March 27, I will give the usual one page quiz. I will also give an optional two page retest, on pages 2 and 4 of this test. If your new score is higher than the original, on one or both pages of the retest, I will replace the original score with the average of the two tries. You should study the maximum and minimum problems in Sections 2.5 and 2.7, together with problems 31–40 in Section 3.1, and problems 41–44 in Section 3.2. Of course, you should also review the rules for differentiation, to make sure that you won’t make any mistakes in using the general power rule or the product rule.