

1. (25 points) Find the derivative of each of these functions.

(a) (5 pts) $f(x) = (e^{x^2} + 5x)^3$ $f'(x) = 3(e^{x^2} + 5x)^2(2xe^{x^2} + 5)$

(b) (5 pts) $f(x) = \ln|x^5 - 3x^4 + 2|$ $f'(x) = \frac{5x^4 - 12x^3}{x^5 - 3x^4 + 2}$

(c) (5 pts) $f(x) = \frac{e^x}{1 + e^x}$ $f'(x) = \frac{e^x(1 + e^x) - e^x(e^x)}{(1 + e^x)^2}$ (using the quotient rule)

(d) (5 pts) $f(x) = \ln\left(\frac{e^x\sqrt{x^2+1}}{(x^2-1)^3}\right) = \ln(e^x) + \ln((x^2+1)^{1/2}) - \ln((x^2-1)^3)$ so

$f(x) = x + (1/2)\ln(x^2+1) - 3\ln(x^2-1)$ and then

$f'(x) = 1 + (1/2)\frac{2x}{x^2+1} - 3\frac{2x}{x^2-1} = 1 + \frac{x}{x^2+1} - \frac{6x}{x^2-1}$

(e) (5 pts) $f(x) = \ln(\ln(x^2)) = \ln(2\ln x)$ $f'(x) = \frac{2/x}{2\ln x} = \frac{1}{x\ln x}$

2. (25 points) (a) (7 pts) $\int (x^4 + 2x - 1) dx = \frac{x^5}{5} + x^2 - x + C$

(b) (7 pts) $\int \left(\frac{1}{x} - \sqrt{x} + e^x\right) dx = \int \left(\frac{1}{x} - x^{1/2} + e^x\right) dx = \ln x - \frac{x^{3/2}}{3/2} + e^x + C = \ln x - \frac{2}{3}x^{3/2} + e^x + C$

(c) (6 pts) Find all functions $f(x)$ with $f'(x) = e^{3x}$ and $f(0) = 1$. Solution: We first find the antiderivative, $f(x) = (1/3)e^{3x} + C$. Then substituting $x = 0$ gives $1 = (1/3)e^0 + C = 1/3 + C$, or $C = 2/3$. The final answer is $f(x) = (1/3)e^{3x} + 2/3$.

(d) (5 pts) Which of the following answers is equal to $\int xe^x dx$?

(i) $\frac{d}{dx}(xe^x + e^x + C) = e^x + xe^x + e^x = xe^x + 2e^x$

(ii) $\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x$

(iii) $\frac{d}{dx}(1/2)x^2e^x + C = xe^x + (1/2)x^2e^x$

The answer is (ii). You need to show the work that justifies your answer.

3. (10 points) Assuming that P_0 and k are constants, differentiate both sides of the equation $y = P_0e^{kt}$ and show that y satisfies the equation $\frac{dy}{dt} = ky$.

$\frac{dy}{dt} = P_0e^{kt}k = yk = ky$ Note: See page 242 of Section 4.3 or page 261 of Chapter 5. This basic idea is critical to know, since it is used in the applications in Chapter 5.

4. (15 points) Let $f(x) = \frac{x}{\ln x}$ be defined for $x > 1$. (a) Show that $f'(x) = \frac{\ln x - 1}{(\ln x)^2}$.

Using the quotient rule, $f'(x) = \frac{(1)(\ln x) - (x)(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{(\ln x)^2}$.

(b) Find the values of x for which $f(x)$ is increasing; find the values of x for which $f(x)$ is decreasing.

We need to find the values of x for which $f'(x)$ is positive, and those for which $f'(x)$ is negative. The sign of $f'(x)$ depends only on the sign of the numerator $\ln x - 1$, and so we first set $\ln x - 1 = 0$. We get $\ln x = 1$, or $e^{\ln x} = e^1$, so $x = e$. Then we can see that $f'(x)$ is negative for $x < e$, and positive for $x > e$, so $f(x)$ is decreasing for $x < e$ and increasing for $x > e$.

(c) The function $f(x)$ has one relative extreme point. Find the coordinates of the point, and determine if it is a relative maximum or a relative minimum.

As before, setting $f'(x) = 0$ gives $x = e$, and then $f(e) = e/\ln(e) = e$. From part (b), since $f(x)$ is decreasing until it reaches $x = e$, and then increasing after that, the point (e, e) must be a relative minimum.

Note: This is similar to problems 26–28 of Section 4.5. Problems 27 and 28 were assigned homework problems.

5. (10 points) A physics professor notices that attendance in his class is decreasing exponentially. After starting with 100 students, there are 90 attending after 4 weeks. How many would he expect to be attending after 12 weeks? (Round your answer to the nearest whole number.)

Since we know what happens every 4 weeks, and 12 is a multiple of 4, we do not need the general formulas. Since the function involves exponential decay, the value will drop by 10% every 4 weeks. After 8 weeks, the attendance would be $(.9) \times 90 = 81$, and after 12 weeks it would be $(.9) \times 81 = 72.9$, or an expected value of 73, when rounded to the nearest whole number.

This can also be solved by setting $P(t) = 100e^{kt}$, and then $90 = P(4) = 100e^{k(4)}$, so $.9 = e^{4k}$, or $k = (1/4)\ln(.9)$. Using this method, the solution is $P(12) = 100e^{3\ln(.9)} = 100(e^{\ln(.9)})^3 = 100(.9)^3 = 72.9$, or an expected attendance of 73.

6. (15 points) The world's population was 5.51 billion on January 1, 1993 and 5.88 billion on January 1, 1998. Assume that at any time the population grows at a rate proportional to the population at that time. In what year will the world's population reach 7 billion?

Since we have exponential growth, we can use the equation $P(t) = 5.51e^{kt}$, where t is the number of years past 1993. The second piece of information is that $P(5) = 5.88$, so $5.51e^{k(5)} = 5.88$, or $5k = \ln(5.88/5.51)$. This gives $k = (1/5)\ln(5.88/5.51)$.

To find when the population is 7 billion we need to solve the equation $P(t) = 7$, or $7 = 5.51e^{kt}$. We get $e^{kt} = 7/5.51$, or $kt = \ln(7/5.51)$, so the solution is $t = (1/k)\ln(7/5.51)$. This value must be added to 1993 to give the actual date.

The curve on this test was 88, 75, 60, 50. There were 8 A's, 11 B's, 20 C's, 16 D's and 20 F's.