1. [p90 #30] $\lim_{x\to 1} \dfrac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x\to 1} \dfrac{\sqrt{x}-(\sqrt{x})^4}{1-\sqrt{x}} = \lim_{x\to 1} \dfrac{\sqrt{x}(1-(\sqrt{x})^3)}{1-\sqrt{x}} = \lim_{x\to 1} \dfrac{\sqrt{x}(1-\sqrt{x})(1+\sqrt{x}+(\sqrt{x})^2)}{1-\sqrt{x}}$
$= \lim_{x\to 1} \sqrt{x}(1+\sqrt{x}+(\sqrt{x})^2) = 1(1+1+1) = 3$ Remember that $x^3-1=(x-1)(x^2+x+1)$.

Another solution: $\lim_{x\to 1} \dfrac{\sqrt{x}-x^2}{1-\sqrt{x}} = \lim_{x\to 1} \dfrac{x^{1/2}(1-x^{3/2})(1+x^{1/2})}{(1-x^{1/2})(1+x^{1/2})} = \lim_{x\to 1} \dfrac{x^{1/2}(1-x^{3/2}+x^{1/2}-x^2)}{1-x}$
$= \lim_{x\to 1} \dfrac{x^{1/2}(1-x^2+x^{1/2}-x^{3/2})}{1-x} = \lim_{x\to 1} \dfrac{x^{1/2}(1-x^2)}{1-x} + \dfrac{x^{1/2}(x^{1/2}-x^{3/2})}{1-x}$
$= \lim_{x\to 1} \dfrac{x^{1/2}(1-x^2)}{1-x} + \dfrac{x-x^2}{1-x} = \lim_{x\to 1} x^{1/2}(1+x)+x = 1(1+1)+1 = 3$

2. [p111 #36] Find the points at which f is discontinuous. $f(x) = \begin{cases} x+1 & \text{if } x\le 1 \\ 1/x & \text{if } 1<x<3 \\ \sqrt{x-3} & \text{if } 3\le x \end{cases}$

As which of these points is f continuous from the right, from the left, or neither?

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} x+1 = 2$ $f(1) = 1+1 = 2$ $\lim_{x\to 1^+} f(x) = \lim_{x\to -1} 1/x = 1$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 1} 1/x = 1/3$ $f(3) = \sqrt{3-3} = 0$ $\lim_{x\to 3^+} f(x) = \lim_{x\to 1} \sqrt{x-3} = 0$

The function f is discontinuous at the points $x=1$ and $x=3$. At $x=1$ it is continuous from the left since $\lim_{x\to 1^-} f(x) = f(1)$. At $x=3$ it is continuous from the right since $\lim_{x\to 3^+} f(x) = f(3)$.

3. [p196 #30] For the equation $\sqrt{x}+\sqrt{y}=1$, find $\dfrac{d^2y}{dx^2}$ by implicit differentiation.

Rewrite the equation as $x^{1/2}+y^{1/2}=1$, then differentiate both sides with respect to x. You must treat y as an *implicit* function of x, so you must use the chain rule.

$\dfrac{1}{2}x^{-1/2}+\dfrac{1}{2}y^{-1/2}\dfrac{dy}{dx}=0$ $x^{-1/2}+y^{-1/2}\dfrac{dy}{dx}=0$ $y^{-1/2}\dfrac{dy}{dx}=-x^{-1/2}$ $\dfrac{dy}{dx}=-y^{1/2}x^{-1/2}$

This gives $\dfrac{dy}{dx} = -\dfrac{\sqrt{y}}{\sqrt{x}}$. To find the second derivative, use $\dfrac{dy}{dx}=-y^{1/2}x^{-1/2}$ and the product rule.

$$\frac{d^2y}{dx^2} = -\frac{1}{2}y^{-1/2}\frac{dy}{dx}\cdot x^{-1/2} - y^{1/2}\cdot\frac{-1}{2}x^{-3/2} = -\frac{1}{2}y^{-1/2}(-y^{1/2}x^{-1/2})x^{-1/2} - y^{1/2}\cdot\frac{-1}{2}x^{-3/2}$$
$$= \frac{1}{2}x^{-1}+\frac{1}{2}y^{1/2}x^{-3/2} = \frac{1}{2}x^{1/2}x^{-3/2}+\frac{1}{2}y^{1/2}x^{-3/2} \qquad \text{Answer: } \frac{d^2y}{dx^2} = \frac{\sqrt{x}+\sqrt{y}}{2x^{3/2}}$$

4. [p380 #14] Find the area of the region bounded by $y=x^3-x$ and $y=3x$.

Solving $x^3-x=3x$ to find the points of intersection, we get $(x+2)(x)(x-2)=0$, so $x=-2,0,2$. Since both functions are odd, the graphs are symmetric about the origin, so we can find the area from $x=0$ to $x=2$ and double it. On this interval, $y=3x$ forms the top of the region, and $y=x^3-x$ forms the bottom (check the relative heights at $x=1$).

$$\text{Area} = 2\int_0^2 (3x-(x^3-x))\,dx = 2\int_0^2 (4x-x^3)\,dx = 2\left(2x^2-\frac{x^4}{4}\right)\Big|_0^2 = 2(8-4) = 8$$

5. [p270] Sketch the graph of $y = \dfrac{x+1}{\sqrt{x^2+1}}$.

Since the denominator is never zero, the graph has no vertical asymptotes. Looking for horizontal asymptotes, we find that $\lim_{x\to\infty} \dfrac{x+1}{\sqrt{x^2+1}} = \lim_{x\to\infty} \dfrac{1+1/x}{\sqrt{1+1/x^2}} = 1$. Similarly, $\lim_{x\to-\infty} \dfrac{x+1}{\sqrt{x^2+1}} = -1$, since the denominator is positive and the numerator is negative. Thus the lines $y=1$ and $y=-1$ are horizontal asymptotes.

To find the first derivative, write $y=(x+1)(x^2+1)^{-1/2}$, and use the product rule.

$$\frac{dy}{dx} = (1)(x^2+1)^{-1/2}+(x+1)(-\tfrac{1}{2})(x^2+1)^{-3/2}(2x) = (x^2+1)^{-1/2}-(x^2+x)(x^2+1)^{-3/2}$$
$$= (x^2+1)(x^2+1)^{-3/2}-(x^2+x)(x^2+1)^{-3/2} = (x^2+1-x^2-x)(x^2+1)^{-3/2} = (1-x)(x^2+1)^{-3/2}$$

Setting $\frac{dy}{dx} = 0$, we get $1 - x = 0$, so $x = 1$ is the only critical number. Substituting $x = 0$ in $\frac{dy}{dx}$ gives a positive value, while substituting $x = 2$ gives a negative value. Conclusion: $\frac{dy}{dx}$ is positive on $(-\infty, 1)$, so y is increasing there; y is decreasing on $(1, \infty)$ since $\frac{dy}{dx}$ is negative; thus y has a relative maximum at $x = 1$.

$$\frac{d^2y}{dx^2} = (-1)(x^2+1)^{-3/2} + (1-x)(-\tfrac{3}{2})(x^2+1)^{-5/2}(2x) = (-1)(x^2+1)(x^2+1)^{-5/2} + (-3x+3x^2)(x^2+1)^{-5/2}$$
$$= (-x^2 - 1 - 3x + 3x^2)(x^2+1)^{-5/2} = (2x^2 - 3x - 1)(x^2+1)^{-5/2}$$

Setting $\frac{d^2y}{dx^2} = 0$, we get $2x^2 - 3x - 1 = 0$. Using the quadratic formula, $x = \frac{3 \pm \sqrt{9 - 4(2)(-1)}}{2 \cdot 2} = \frac{3 \pm \sqrt{17}}{4}$. The approximate solutions are $x = -.28$ and $x = 1.78$. Substituting $x = -1$, $x = 0$, and $x = 2$ into $\frac{d^2y}{dx^2}$ gives a numerator of 4, -1, and 1, respectively. Conclusion: y is concave up on $(-\infty, -.28)$ and $(1.78, +\infty)$, and concave down on $(-.28, 1.78)$; there are points of inflection when $x = \frac{3 \pm \sqrt{17}}{4}$.

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1. [p144] Use the definition of the derivative to find $f'(x)$, for $f(x) = \frac{x+1}{x-1}$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h)+1}{(x+h)-1} - \frac{x+1}{x-1}\right] \qquad \text{substitute}$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x+h+1)(x-1) - (x+h-1)(x+1)}{(x+h-1)(x-1)}\right] \qquad \text{find a common denominator}$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1) - (x^2+hx-x+x+h-1)}{(x+h-1)(x-1)}\right] \qquad \text{expand}$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2h}{(x+h-1)(x-1)}\right] = \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)} \qquad \text{simplify and cancel } h$$
$$= \frac{-2}{(x-1)^2} \qquad \text{take the limit}$$

Check using the quotient rule: $f'(x) = \frac{(1)(x-1) - (x+1)(1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$

2. [p155] Find $\frac{ds}{dt}$ for $x = \sqrt{t}(t^3 - \sqrt{t} + 1)$. Check to see if you can first simplify the function algebraically.

$$\frac{d}{dt}(\sqrt{t}(t^3 - \sqrt{t} + 1)) = \frac{d}{dt}(t^{1/2}(t^3 - t^{1/2} + 1)) = \frac{d}{dt}(t^{7/2} - t + t^{1/2}) = \tfrac{7}{2}t^{5/2} - 1 + \tfrac{1}{2}t^{-1/2}$$

3. [p182 #42] Find the derivative of $y = \sqrt{\cos(\sin^2 x)}$.

$$\frac{d}{dx}\sqrt{\cos(\sin^2 x)} = \frac{d}{dx}(\cos((\sin x)^2))^{1/2} = \tfrac{1}{2}(\cos((\sin x)^2))^{-1/2}(-\sin((\sin x)^2))(2\sin x \cos x)$$

4. [p284 #11] If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Let the height be h and the length of the base be x. We need to maximize the volume $V = x^2h$. The constraint equation is $x^2 + 4xh = 1200$, since the area of the base plus the area of the four sides must equal the total amount of material. After solving the constraint equation for h and substituting into the formula for the volume, we get the following equations.

$$V(x) = x^2h = x^2\left(\frac{1200 - x^2}{4x}\right) = 300x - \tfrac{1}{4}x^3 \qquad V'(x) = 300 - \tfrac{3}{4}x^2 \qquad V''(x) = -\tfrac{3}{2}x$$

Setting $V'(x) = 0$, we get $300 - \frac{3}{4}x^2 = 0$ or $x^2 = 400$, so $x = \pm 20$. In the problem, the solution must be positive, so $x = 20$, and then $h = (1200 - 400)/4(20) = 10$. Finally, we use the second derivative test: $V''(x) = -\frac{3}{2}x$, so $V''(20) = -30$, and therefore $V(x)$ is concave down at $x = 20$. This shows that choosing $x = 20$ does indeed maximize the volume. Answer: the maximum volume is $V(20) = (20)^2 \cdot 10 = 4000$ cm^3.

5. [p307 #58] A particle is moving with acceleration $a(t) = 10 + 3t - 3t^2$, given $s(0) = 0$ and $s(2) = 10$.

We need to take the antiderivative of $a(t)$ to find the velocity $v(t)$, and then the position $s(t)$ is given by the antiderivative of $v(t)$. $v(t) = 10t + \frac{3}{2}t^2 - t^3 + C$ $\quad s(t) = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 + Ct + D$ $\quad s(0) = 0$, so $D = 0$

When $t = 2$, $s = 10$, so $10 = 5(2)^2 + \frac{1}{2}(2)^3 - \frac{1}{4}(2)^4 + 2C$ or $10 = 20 + 4 - 4 + 2C$, so $C = -5$.

Answer: the position at time t is given by the formula $s(t) = 5t^2 + \frac{1}{2}t^3 - \frac{1}{4}t^4 - 5t$.