Review of differentiation

We had to use a completely new idea (that of a limit) to find the rate of growth of a function. For a function $y = f(x)$ we defined $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, and then we were able to find some basic formulas: if $f(x) = x^n$, then $f'(x) = nx^{n-1}$; if $f(x) = \sin x$, then $f'(x) = \cos x$. With these formulas we can differentiate more complicated functions with the help of the product rule, quotient rule, and chain rule.

The first derivative can be used to find the local maximum and local minimum values of a function, while the second derivative measures the curvature of its graph. This helps in graphing (we can focus on plotting the most important points of the graph) and allows us to solve applied maximum and minimum problems.

Motivating integration

The formula *rate* $\times$ *time* = *distance* works whenever the rate is constant. This formula can be applied to many other situations when we know the rate at which something occurs. For instance, if we know the rate at which a factory is using electricity, then we can calculate the total used in a given time. But the formula fails to work if the rate is variable. We need to replace it with the formula *average rate* $\times$ *time* = *total output*.

There are similar formulas for area and volume. The area of a rectangle is found by multiplying *height* $\times$ *width*. But suppose that the height varies. We want to be able to deal with the situation in which the top of the rectangle is replaced by a curve whose formula we know. What we need is the formula *area* = *average height* $\times$ *width*.

As a final example, for a box or for a right circular cylinder we can calculate the volume using the formula *volume* = *area of the base* $\times$ *height*. This works because every cross section of the solid is exactly the same as the base. In a cone or a pyramid, this is no longer true. In a right circular cone (with its pointed end up) the cross section areas vary from πr^2 for the circle at the bottom to 0 at the point. But we could still calculate the volume if we had some way to find the *average cross section area*, which we could multiply by the *height*.

The meaning of the definite integral

The notation $\int_a^b f(x)\,dx$ is used when we calculate the average height of $f(x)$ on the interval $[a, b]$ and multiply it by $(b - a)$. This is used in the problems described above. If $f(x)$ is the function which describes the rate at which something is happening, then $\int_a^b f(x)\,dx$ gives the total output from time $x = a$ to time $x = b$.

Let's say we have a rectangle, with its base on the x-axis, its left hand side along the vertical line $x = a$ and its right hand side along the vertical line $x = b$. Instead of a horizontal line at the top, let's say that the top is given by the graph of $y = f(x)$. Then the area of the rectangle is given by $\int_a^b f(x)\,dx$. This construction is usually described by saying that $\int_a^b f(x)\,dx$ gives the area under the curve $y = f(x)$, from $x = a$ to $x = b$.

Calculating definite integrals

This can be pretty hard, and will take a lot of time to develop. For difficult functions we are forced to use approximations and then find a limit. Note: your calculator can do the approximations, and has a way of computing definite integrals.

Here is one situation in which we *can* give a formula for evaluating a definite integral. Suppose we are able to find a function $F(x)$ whose derivative is $f(x)$. (For some functions this is easy to do; for others it is impossible to find a "nice" formula for $F(x)$.) It turns out that finding the average of instantaneous rates of change is easy.

Think of it this way. Suppose you drive on I-88 for an hour. That would make it easy to calculate your average speed. Then suppose that you had some way to record your speedometer readings at each instant. If there is a good way to average the speedometer readings, then surely it has to just give you the same answer: your average speed.

If $f(x)$ is the derivative of $F(x)$, then it gives the instantaneous rate of growth of $F(x)$, If we average these "speedometer" readings for $F(x)$, starting with $x = a$ and ending with $x = b$, we should just get the average rate of change $\dfrac{F(b) - F(a)}{b - a}$ of $F(x)$ over the interval $[a, b]$. Then we need to multiply by $(b - a)$, so $\int_a^b f(x)\,dx = \dfrac{F(b) - F(a)}{(b - a)} \cdot (b - a) = F(b) - F(a)$. This idea is important enough to be called a theorem.

[Fundamental Theorem of Calculus] If $f(x) = F'(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

The function $F(x)$ is called an *antiderivative* for $f(x)$. To work with more complicated functions, we will need to figure out how to use the product rule and chain rule in reverse. But for now you can do problems like these.

1. $\int_1^4 (3x^2 - 5)\, dx =$

The first step is to find an antiderivative for $3x^2 - 5$. Make an educated guess: when you take the derivative of a polynomial, the degree goes down. So to find an antiderivative we need to raise the degree. In fact, $\dfrac{d}{dx}(x^3 - 5x) = 3x^2 - 5$, so we can take $F(x) = x^3 - 5x$. Here are the steps that are usually used.

$$\int_1^4 (3x^2 - 5)\, dx = x^3 - 5x\Big|_1^4 = (4^3 - 5 \cdot 4) - (1^3 - 5 \cdot 1) = 64 - 20 - 1 + 5 = 69 - 21 = 48$$

2. $\int_0^\pi \sin x\, dx = -\cos x\Big|_0^\pi = -\cos \pi - (-\cos 0) = -(-1) - (-1) = 2.$

3. $\int_0^{2\pi} \sin x\, dx = 0$ Why? Between 0 and 2π the average height of the sine function is 0.

4. Verify the formula $V = \frac{1}{3}\pi r^2 h$ for the volume of a right circular cone.

Visualize the cone lying on its side, with the point at the origin and the x-axis running down through the center. If you slice the cone perpendicular to the axis, at the point x, you have a cross-section that is a circle of radius y and an area of πy^2. The value of y that corresponds to x is found by looking at similar triangles. We yet $\dfrac{y}{x} = \dfrac{r}{h}$, so that $y = \dfrac{r}{h}x$. Thus the cross-section area at x is $\pi y^2 = \dfrac{\pi r^2 x^2}{h^2}$, so to find the average cross-section area times the height we get $\int_0^h \dfrac{\pi r^2}{h^2} x^2\, dx = \dfrac{\pi r^2}{h^2}\dfrac{x^3}{3}\Big|_0^h = \dfrac{\pi r^2}{h^2}\dfrac{h^3}{3} = \frac{1}{3}\pi r^2 h.$