

Exam 3 will cover Sections 4.7, 4.9, 4.10, 5.3, 5.4, and 5.5. You will not be allowed to use a graphing calculator on the exam. The test will be designed so that all calculations can be done by hand.

Before beginning the review, I need to recall the terminology of *dependent* and *independent* variables (for references in the text, see page 12 for functions of one variable and page 923 for functions of two variables). If we write  $y$  as a function of  $x$ , say  $y = f(x)$ , then we consider  $y$  to be dependent on  $x$ , and so  $y$  is called the *dependent* variable. Because we think of  $x$  as the variable to which we can assign any value we want,  $x$  is called the *independent* variable.

In some functions there may be more than one independent variable. For example, if  $V$  is the volume of a box with a square base, then can write  $V = x^2h$ , where  $x$  is the width of the base, and  $h$  is the height of the box. Here  $V$  is a dependent variable, and  $x$  and  $h$  are independent variables, because we have expressed the volume in terms of the width and height, and these can be varied independently.

#### Optimization (4.7)

Some helpful steps are listed in the text on page 278, but to organize your work I suggest that you follow the steps I have listed below. First, just *scan* the problem to get the answers to step 1 and step 2. Then go through the problem *again*, carefully, to assign names to the variables and to find the equations that are involved.

1. Write down what it is that you are asked to maximize or minimize. This is the *dependent* variable.
2. Find the corresponding independent variable (or variables). That is, decide how the dependent variable can be changed, and which quantities can be used to make this change. If you don't see this relationship right away, try some examples, using easy numbers.
3. If you seem to have more than one independent variable, then there must be a *constraint equation* that relates them, because at this point we only know how to optimize a function of one variable. Find the constraint equation, but *keep it separate* from the formula for the dependent variable, so that you do *not* optimize the wrong function. You might even want to draw a line down the middle of the page, putting your formula for the dependent variable on one side, and putting the constraint equation, diagrams, etc. on the other side.
4. Find a formula for the dependent variable, set its derivative equal to zero, and solve.
5. Check that you really did find the maximum or minimum by using either the first derivative test or the second derivative test.

#### Newton's method (4.9)

Given an approximate solution  $x_n$  to the equation  $f(x) = 0$ , Newton's method uses the tangent line at  $(x_n, f(x_n))$  to solve for the next approximation. The tangent line is  $y = f'(x_n)(x - x_n) + f(x_n)$ . Setting  $y = 0$  and solving for  $x$  gives the general formula  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ . As an example, the formula to approximate  $\sqrt{a}$  is  $x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right)$ .

#### Definite integrals

The definite integral  $\int_a^b f(x) dx$  can be interpreted as the average height of  $f(x)$  on  $[a, b]$  multiplied by the length  $(b - a)$  of the interval. In the text, this is not mentioned until Section 6.5 (page 403), where you will find the formula  $f_{ave} = \frac{1}{b - a} \int_a^b f(x) dx$ . I prefer to write this as

$$\int_a^b f(x) dx = (f_{ave}) \cdot (b - a) .$$

If the graph of  $f(x)$  lies above the axis on the interval  $[a, b]$ , then  $\int_a^b f(x) dx$  is the average height  $f_{ave}$  multiplied by the width  $(b - a)$ . That explains why the definite integral gives us the area under the curve  $y = f(x)$ , from  $x = a$  to  $x = b$ .

As another application, when an object is traveling at a constant velocity  $v$ , we have the familiar formula  $d = v \cdot t$ , where  $d$  is the distance, and  $t$  is the time. When the velocity is not constant, but we know  $v$  as a function of  $t$ , we can find the distance by multiplying the *average* velocity by the time. This gives us the distance formula  $d = \int_a^b v(t) dt$ .

Why do we use integrals instead of just finding averages of functions? The answer is that the integral is what you need in most applications, and its properties can be expressed in nicer formulas.

The Fundamental Theorem of Calculus (5.3)

There are times when the definite integral can only be evaluated by approximations using Riemann sums. But integrals would not have proved to be so useful without the Fundamental Theorem of Calculus.

**Theorem.** (page 344) If  $f$  is a continuous function defined on  $[a, b]$ , and  $F$  is any antiderivative of  $f$ , then

$$\int_a^b f(x) dx = F(b) - F(a).$$

I like to explain the Fundamental Theorem this way. If  $f(x)$  is the derivative of  $F(x)$ , then it represents the instantaneous rate of change of  $F(x)$ . If we average all of these instantaneous rates of change, we should get the average rate of change of  $F(x)$ . Using our notation for averages, this says that

$$F'(x)_{ave} = \frac{F(b) - F(a)}{b - a} \quad \text{or} \quad (F'(x)_{ave}) \cdot (b - a) = F(b) - F(a),$$

which gives us the formula  $\int_a^b F'(x) dx = F(b) - F(a)$ . If we *didn't* get this connection between averages and rates of change, something would be wrong with the definitions.

Antiderivatives (4.10, 5.4)

The Fundamental Theorem of Calculus has an equivalent statement (see page 342):

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

With this in mind, we use the *indefinite* integral  $\int f(x) dx$  to denote the general antiderivative of  $f$ . Each differentiation formula has a corresponding integration formula (see page 351 for a list that updates the formulas for antiderivatives given on page 301). For example,  $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ , for all integers  $n \neq -1$ .

Integration by substitution (5.5)

The integration technique that corresponds to the *chain rule* is called *integration by substitution*. This technique uses the chain rule in reverse. The chain rule for differentiation can be written in the form

$$\frac{d}{dx} F(u(x)) = F'(u(x))u'(x).$$

If we let  $F'(x) = f(x)$ , then the corresponding integration formula is

$$\int f(u(x))u'(x)dx = F(u(x)) + C.$$

In an integration problem, sometimes it is hard to see how to use a composite function, so we can choose  $u(x)$  and then make a substitution. The best choice for  $u(x)$  is usually a quantity in parentheses. We also need to replace  $u'(x)dx$  with  $du$ . Remember the formula for the differential of  $u$ :  $du = u'(x)dx$ . I find it best to solve for  $dx$ , and then substitute  $\frac{du}{u'(x)}$  for  $dx$  in the original integral. Substituting for  $u$  and for  $dx$ , in this way, should eliminate all of the  $x$  terms. If not, you need to choose a different  $u(x)$  and try again.

In a definite integral we can also substitute for the limits of integration, and so we get the formula on page 363:

$$\int_a^b f(u(x))u'(x) dx = \int_{u(a)}^{u(b)} f(u) du.$$

In more detail,  $\int_{x=a}^{x=b} f(u(x))u'(x) dx = \int_{u=c}^{u=d} f(u) du$ , where  $c = u(a)$  and  $d = u(b)$  are the new limits.

## Review problems

After you have reviewed all of the assigned homework problems, and the quizzes, try these problems from some old tests.

1. (a)  $\int_1^2 (1 - 2x - 3x^2) dx =$   
 (b)  $\int_0^1 \frac{x^2 + 1}{\sqrt{x}} dx =$   
 (c)  $\int \frac{x^4 - 1}{x^2 + 1} dx =$   
 (d)  $\int \frac{\sin x}{1 - \sin^2 x} dx =$
2. Use the Fundamental Theorem of Calculus to find the derivative:  $\frac{d}{dx} \int_3^{x^2} \frac{\cos(t)}{2t} dt =$
3. Find  $f(x)$ , given that  $f''(x) = 12x^2 + 6x$  and that the graph of  $y = f(x)$  passes through the point  $(-1, -1)$ , where it has a horizontal tangent line.
4. Use differentiation to verify that this integration formula is correct.  $\int \frac{x}{\sqrt{x^2 + 1}} dx = \sqrt{x^2 + 1} + C$
5. Evaluate this integral by interpreting it as an area.  $\int_{-1}^1 (2 - \sqrt{1 - x^2}) dx$
6. A farmer has 2400 feet of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimension of the field that has the largest area?
7. A box with a square base and an open top is to hold 32 cubic inches. Find the dimensions of the box that minimize the amount of material used. (Neglect the thickness of material and any waste in construction.) Check your answer by using the first or second derivative test.
8. The product of two positive numbers is 9. What is the smallest possible value of the sum of their squares?
9. (a) State the formula used for Newton's method.  
 (b) Use Newton's method to approximate the root of  $f(x) = x^3 - 3x + 1$  between 0 and 1. Start with  $x_1 = 0$ , and calculate  $x_2$  and  $x_3$  (as fractions).  
 (c) Explain why Newton's method *doesn't* work if the initial approximation is chosen to be  $x_1 = 1$ .
10. (a)  $\int 21x^2(x^3 + 7)^5 dx =$   
 (b)  $\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx =$   
 (c)  $\int_{-1}^0 x^2 \sqrt{x^3 + 1} dx =$
11. What is the area of the region above the x-axis and under the graph of  $y = 6x^2 + 1$  over the interval  $[-1, 2]$ ?
12. Verify the integration formula  $\int x \sin x dx = \sin x - x \cos x + C$ .

Solutions to some of the review problems

1. (a)  $\int_1^2 (1 - 2x - 3x^2) dx = x - x^2 - x^3 \Big|_1^2 = (2 - 4 - 8) - (1 - 1 - 1) = -9$

(b)  $\int_0^1 \frac{x^2 + 1}{\sqrt{x}} dx = \int_0^1 (x^2 + 1)(x^{-1/2}) dx = \int_0^1 (x^{3/2} + x^{-1/2}) dx = \frac{2}{5}x^{5/2} + 2x^{1/2} \Big|_0^1 = \frac{2}{5} + 2 = 2\frac{2}{5}$

(c)  $\int \frac{x^4 - 1}{x^2 + 1} dx = \int \frac{(x^2 - 1)(x^2 + 1)}{x^2 + 1} dx = \int (x^2 - 1) dx = \frac{x^3}{3} - x + C$

(d)  $\int \frac{\sin x}{1 - \sin^2 x} dx = \int \frac{\sin x}{\cos^2 x} dx = \int \frac{1}{\cos x} \frac{\sin x}{\cos x} dx = \int \sec x \tan x dx = \sec x + C$

2.  $\frac{d}{dx} \int_3^{x^2} \frac{\cos(t)}{2t} dt = \frac{\cos(x^2)}{2x^2} (2x) = \frac{\cos(x^2)}{x}$  (substitute  $x^2$  for  $t$  and multiply by the derivative of  $x^2$ )

3.  $f''(x) = 12x^2 + 6x$        $f'(x) = 4x^3 + 3x^2 + C$        $f'(-1) = 0$  so  $0 = -4 + 3 + C$  and  $C = 1$   
 $f'(x) = 4x^3 + 3x^2 + 1$        $f(x) = x^4 + x^3 + x + D$        $f(-1) = -1$  so  $-1 = 1 - 1 - 1 + D$  and  $D = 0$   
 Answer:  $f(x) = x^4 + x^3 + x$

4.  $\int \frac{x}{\sqrt{x^2 + 1}} dx = \sqrt{x^2 + 1} + C$       since       $\frac{d}{dx}(x^2 + 1)^{1/2} = \frac{1}{2}(x^2 + 1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + 1}}$

5.  $\int_{-1}^1 (2 - \sqrt{1 - x^2}) dx$        $y = 2 - \sqrt{1 - x^2}$ , so  $(y - 2)^2 = (\sqrt{1 - x^2})^2$ , or  $x^2 + (y - 2)^2 = 1$

The graph is the bottom half of the circle with radius 1 and center  $(2, 0)$ . The area beneath this curve

(from  $x = -1$  to  $x = 1$ ) is  $4 - \pi/2$ , so  $\int_{-1}^1 (2 - \sqrt{1 - x^2}) dx = 4 - \frac{\pi}{2}$

6. See Example 1 on page 278 of the text.

7. Assume that the box is  $x$  by  $x$  by  $y$ . We need to maximize the surface area  $A = x^2 + 4xy$ , given that the volume is  $x^2y = 32$ . Solve for  $y$  and substitute in  $A$  to get  $A(x) = x^2 + 128x^{-1}$ . Then  $A'(x) = 2x - 128x^{-2}$  and  $A''(x) = 2 + 256x^{-3}$ . Setting  $A'(x) = 0$  gives  $x^3 = 64$ , so  $x = 4$ . Then  $y = 2$ , and  $A''(4) = 6$ , so  $A(x)$  is concave up at  $x = 4$  and we *have* found a minimum value for the surface area.

8. Let  $x$  and  $y$  be the positive numbers. We need to minimize  $f(x, y) = x^2 + y^2$ . The constraint is that  $xy = 9$ , where  $x > 0$  and  $y > 0$ . Solving for  $y$  in the constraint equation we get  $y = 9/x$ , and substituting into  $f(x, y)$  gives us  $f(x) = x^2 + \frac{81}{x^2} = x^2 + 81x^{-2}$ . Now we can differentiate and set the derivative equal to zero, giving  $0 = 2x - (2)(81)x^{-3}$ . We get  $x^4 = 81$ , so  $x = \pm 3$ . The negative solution is ruled out, so  $x = 3$  and  $y = 9/3 = 3$ . Finally,  $f''(x) = 2 + (6)(81)x^{-4}$  and  $f''(3)$  is positive, so we *did* find the minimum value for  $x^2 + y^2$ .

9. The formula we need is  $x_{n+1} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}$ . If  $x_1 = 0$ , then  $x_2 = 0 - \frac{1}{-3} = \frac{1}{3}$ . Next,

$$x_3 = \frac{1}{3} - \frac{(\frac{1}{3})^3 - 3(\frac{1}{3}) + 1}{3(\frac{1}{3})^2 - 3} = \frac{1}{3} - \frac{(\frac{1}{3})^3 - 1 + 1}{\frac{1}{3} - 3} = \frac{1}{3} - \frac{(\frac{1}{3})^3 - 1}{\frac{1}{3} - 3} = \frac{1}{3} - \frac{(\frac{1}{3})^3 - 1}{\frac{1}{3} - 3} = \frac{1}{3} - \frac{(\frac{1}{3})^3 - 1}{\frac{1}{3} - 3} = \frac{1}{3} - \frac{(\frac{1}{3})^3 - 1}{\frac{1}{3} - 3} = \frac{24}{72} + \frac{1}{9} \cdot \frac{1}{8} = \frac{25}{72}.$$

10. (a) Let  $u = x^3 + 7$ , so  $du = 3x^2 dx$ , or  $x^2 dx = \frac{du}{3}$ . Substituting for  $x^3 + 7$  and  $x^2 dx$  gives

$$\int 21x^2(x^3 + 7)^5 dx = \int 21(x^3 + 7)^5 x^2 dx = \int 21u^5 \frac{du}{3} = 7 \int u^5 du = 7 \frac{u^6}{6} + C = \frac{7}{6}(x^3 + 7)^6 + C$$

(b) Let  $u = \sqrt{x} = x^{1/2}$ , so  $du = \frac{1}{2}x^{-1/2} dx$ , or  $\frac{dx}{\sqrt{x}} = 2du$ . Then

$$\int \frac{\sin(\sqrt{x})}{\sqrt{x}} dx = \int \sin(u) 2du = -2 \cos(u) + C = -2 \cos(\sqrt{x}) + C$$

(c) Let  $u = x^3 + 1$ , so  $du = 3x^2 dx$  and  $x^2 dx = \frac{du}{3}$ . When  $x = -1$ ,  $u = 0$  and when  $x = 0$ ,  $u = 1$ .

$$\int_{-1}^0 x^2 \sqrt{x^3 + 1} dx = \int_{x=-1}^{x=0} \sqrt{x^3 + 1} x^2 dx = \int_{u=0}^{u=1} \sqrt{u} \frac{du}{3} = \frac{1}{3} \int_0^1 u^{1/2} du = \frac{1}{3} \cdot \frac{2}{3} u^{3/2} \Big|_0^1 = \frac{2}{9}(1)^{3/2} - \frac{2}{9}(0)^{3/2} = \frac{2}{9}$$