

The final exam is comprehensive. You will not be allowed to use a graphing calculator on the exam. The test will be designed so that all calculations can be done by hand.

Approximating integrals using Riemann sums (5.1, 5.2)

Definition. (page 324) If f is a continuous function defined on $[a, b]$, then

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \frac{(b-a)}{n},$$

where the interval $[a, b]$ is divided into n subintervals of equal length, and the points x_i^* are chosen in the i th subinterval.

This limit can be written in the form $\int_a^b f(x) dx = \left(\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{f(x_i^*)}{n} \right) \cdot (b-a)$, which shows that it can be interpreted as the limit of the average of a finite number of sample points on the curve, multiplied by $(b-a)$. Of course, you also need to remember that the definite integral can be interpreted as a limit of Riemann sums (see page 325). When you find the area under a curve, these Riemann sums represent the area found by using approximating rectangles. See pages 333 to 335 for lists of the important properties of the definite integral.

Given a function $f(x)$, you need to know how to set up the Riemann sum that will evaluate the integral $\int_a^b f(x) dx$. You also need to know how to do the opposite problem: given a Riemann sum, what integral does it represent? Questions of the following type often show up on the final exam.

1. Use a Riemann sum to estimate the area under the graph of $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 5$, using four equal subintervals and the left hand endpoints of the subintervals.

Answer: The Riemann sum is $1 + 1/2 + 1/3 + 1/4 = (12 + 6 + 4 + 3)/12 = 25/12$

2. Use the midpoint rule, with 4 subdivisions, to approximate the integral $\int_1^5 \frac{x-1}{x+1} dx$. Use fractions, but leave

your answer as the sum of 4 fractions. Answer: $\int_1^5 \frac{x-1}{x+1} dx \approx \frac{1}{5} + \frac{3}{5} + \frac{5}{9} + \frac{7}{11}$.

The area between two curves (6.1)

If $f(x) \geq 0$, then we have used the definite integral $\int_a^b f(x) dx$ to find the area between $f(x)$ and the x -axis. This works because the integral multiplies the average height by the width. The top of the region follows a curve, while the bottom is a straight line.

In this section we allow both the top and the bottom of the region to be given by a curve. If $f(x) \geq g(x)$, then the area below $f(x)$ and above $g(x)$, from $x = a$ to $x = b$, is given by the integral $\int_a^b [f(x) - g(x)] dx$. Again, this works because the integral gives the average height times the width.

If we don't know whether or not $f(x) \geq g(x)$, we can use $\int_a^b |f(x) - g(x)| dx$ to give the area. To actually evaluate this integral, we need to split the interval $[a, b]$ up into intervals on which either $f(x) \geq g(x)$ or $g(x) \geq f(x)$. To make sure that we get a positive answer for the area, we always need to integrate the top function minus the bottom function. If the functions are at all complicated, it helps to graph them before trying to integrate.

Sometimes it is best to take the average width times the height. If the right hand side is given by $x = f(y)$, and the left hand side by $x = g(y)$, then the area is $\int_{y=a}^{y=b} [f(y) - g(y)] dy$.