1. (20 points) Find the derivative \( f'(x) \) or \( \frac{dy}{dx} \).

(a) (p157 #14) \( f(x) = \sqrt{x} - \frac{1}{\sqrt{x}} = x^{1/2} - x^{-1/2} \) \( f'(x) = (1/2)x^{-1/2} + (1/2)x^{-3/2} \)

(b) (p183 #17) \( f(x) = (3x - 2)^{10}(5x^2 - x + 1)^{12} \) Use the product rule
\( f'(x) = (10)(3x - 2)^9(3)(5x^2 - x + 1)^{12} + (3x - 2)^{10}(12)(5x^2 - x + 1)^{11}(10x - 1) \)

(c) (p215 #31) \( y = \sin(\tan(\sqrt{1 + x^2})) \) Use the chain rule (3 times)
\( \frac{dy}{dx} = \cos(\tan(\sqrt{1 + x^2})) \cdot (\sec^2(\sqrt{1 + x^2})) \cdot (1/2)(1 + x^2)^{-1/2}(3x^2) \)

(d) (p187 Ex 2) \( x^3 + y^3 = 6xy \) Use implicit differentiation (See the text for the solution)

2. (5 points; p197 #17) Find the second derivative \( f''(x) \) for \( f(x) = \tan(3x) \). Use the chain rule.
\( f'(x) = (\sec^2(3x))(3) = 3(\sec(3x))^2 \)
\( f''(x) = 6(\sec(3x)) \cdot \sec(3x) \cdot \tan(3x) \cdot (3) \)

3. (6 points; p176 #40) \( \lim_{x \to 0} \frac{\tan x}{4x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{4 \cos x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0} \frac{1}{4 \cos x} = 1 \cdot \frac{1}{4 \cdot 1} = \frac{1}{4} \)

4. (7 points; p157 #64) Find the equations of the tangent lines to the curve \( y = \frac{x - 1}{x + 1} \) that are parallel to the line \( x - 2y = 2 \).

\[ x - 2y = 2 \quad 2y = x - 2 \quad y = \frac{1}{2}x - 1 \]

We need to solve \( y' = \frac{1}{2} \) Using the quotient rule, we get
\( y' = \frac{1(x + 1) - (x - 1)(1)}{(x + 1)^2} = \frac{x + 1 - x + 1}{(x + 1)^2} = \frac{2}{(x + 1)^2} \)

Setting \( \frac{2}{(x + 1)^2} = \frac{1}{2} \), we get \( 4 = (x + 1)^2 \), so \( x + 1 = \pm 2 \), or \( x = 1, -3 \). When \( x = 1 \), \( y = 0 \), and the corresponding tangent line is \( y = \frac{1}{2}(x - 1) \). When \( x = -3 \), \( y = 2 \), and the corresponding tangent line is \( y = \frac{1}{2}(x + 3) + 2 \).

5. (6 points; p211 #24) For \( y = \sqrt{1 - x} \), find the differential \( dy \) and evaluate \( dy \) for \( x = 0 \) and \( dx = 0.02 \).
\( \frac{dy}{dx} = \frac{1}{2}(1 - x)^{-1/2} = \frac{1}{2\sqrt{1 - x}} \). When \( x = 0 \), \( \frac{dy}{dx} = \frac{1}{2} \) and so \( dy = \frac{dy}{dx} dx = \frac{1}{2} \cdot 0.02 = 0.01 \).

6. (6 points; p217 #79) A window has the shape of a square surmounted by a semicircle. The base of the window is measured as having width 60cm, with a possible error of 0.1cm. Use differentials to estimate the maximum error possible in computing the area of the window.

Let \( x \) be the width of the window. Then the area of the window is \( A(x) = x^2 + \frac{1}{2} \pi \left( \frac{x}{2} \right)^2 = x^2 + \frac{1}{8} \pi x^2 \), and \( \Delta A = A'(x) \Delta x = (60^2 + \frac{1}{8} \pi (60^2))(0.1) = (3600 + \pi(15)(30))(0.1) = (3600 + 450\pi)(0.1) = 360 + 450\pi \).

7. (8 pts; p167 #8) If a ball is thrown vertically upward with a velocity of 80 ft/sec, then its height after \( t \) seconds is \( s = 80t - 16t^2 \).
(a) What is the maximum height reached by the ball?

We have \( s'(t) = 80 - 32t \). Set \( s'(t) = 0 \), so \( 80 - 32t = 0 \), or \( t = 2.5 \). Then the corresponding height is \( s = 80(2.5) - 16(6.25) = 200 + 100 = 300 \text{ ft} \).

(b) What is the velocity of the ball when it is 96 ft above the ground on the way up?

When \( s = 96 \), we get \( 96 = 80t - 16t^2 \) or \( 16t^2 - 80t + 96 = 0 \). Dividing by 16 gives \( t^2 - 5t + 6 = 0 \), so \( t = 2 \) or \( t = 3 \). The first solution must correspond to a height of 96 ft on the way up. Substituting \( t = 2 \) into the formula for the derivative gives \( s'(2) = 80 - 64 = 16 \), so the velocity is 16ft/sec.
8. (17 pts; p248 #32) For the function \( f(x) = x^3 - 12x + 1 \), graph the function after finding
(a) the intervals on which \( f \) is increasing or decreasing:

We need to analyze the sign of \( f'(x) = 3x^2 - 12 \). We have \( f'(x) = 3(x^2 - 4) = 3(x + 2)(x - 2) \), so \( f'(x) = 0 \) when \( x = -2 \) or \( x = 2 \). We need to look at the sign of \( f'(x) \) on the intervals \((−∞, -2)\), \((-2, 2)\), and \((2, +∞)\). Test at \( x = -3, x = 0, \) and \( x = 3 \). We get \( f'(-3) = (3)(-1)(-5) \), which is positive, \( f'(0) = (3)(2)(-2) \), which is negative, and \( f'(3) = (3)(5)(1) \), which is positive.

Conclusion: \( f(x) \) is increasing on \((−∞, -2)\) and \((2, +∞)\) and decreasing on \((-2, 2)\).

(b) the local maximum and minimum values of \( f \);

Using the first derivative test we can see that there is a local maximum at \( x = -2 \) and a local minimum at \( x = 2 \). The corresponding points on the curve are \((-2, 17)\) and \((2, -15)\), since \( f(-2) = -8 + 24 + 1 = 17 \) and \( f(2) = 8 - 24 + 1 = -15 \).

(c) the intervals of concavity and the inflection points.

We need to analyze the sign of \( f''(x) = 6x \). The graph is concave down on \((−∞, 0)\), and concave up on \((0, +∞)\), so \((0, 1)\) is the inflection point.

9. (13 points; p248 #32) For the function \( f(x) = (x^2 - 1)^3 \), find
(a) the intervals on which \( f \) is increasing or decreasing:

We have \( f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2 \). Setting \( f'(x) = 0 \) we get \( 6x(x^2 - 1)^2 = 0 \), so either \( x = 0 \) or \( x^2 - 1 = 0 \), giving \( x = ±1 \). We need to test the sign of \( f'(x) \) on the intervals \((−∞, -1)\), \((-1, 0)\), \((0, 1)\), and \((1, +∞)\).

Instead of choosing a point in each interval, it is probably better to analyze the sign of each factor of \( f'(x) \). Since \( f'(x) = 6x(x^2 - 1)^2 \), look at the factors \( 6x \) and \( x^2 - 1 \) separately. The first factor is negative for \( x < 0 \) and positive for \( x > 0 \). The second factor is never negative. Conclusion: \( f'(x) ≤ 0 \) for \( x < 0 \) and \( f'(x) ≤ 0 \) for \( x > 0 \), so \( f(x) \) is increasing on \((−∞, -1)\) and \((-1, 0)\) and decreasing on the intervals \((0, 1)\) and \((1, +∞)\).

(b) the local maximum and minimum values of \( f \);

Although \( f'(x) = 0 \) for \( x = -1, 0, 1 \), the derivative does not change sign at \( x = -1 \) and \( x = 1 \), so these points are neither a relative max nor a relative min. At \( x = 0 \), the function changes from decreasing to increasing, so there is a relative minimum at \((0, -1)\). (Note that \( f(0) = (0^2 - 1)^3 = -1 \).

(c) the intervals of concavity and the inflection points.

Since \( f'(x) = 6x(x^2 - 1)^2 \), we need to use the product rule to find \( f''(x) \). We get \( f''(x) = 6(x^2 - 1)^2 + (6x)(2)(x^2 - 1)(2x) = (x^2 - 1)(6x^2 - 6 + 24x^2) = (x^2 - 1)(30x^2 - 6) = 6(x^2 - 1)(5x^2 - 1) \).

Again, we could test values in each of the 5 intervals that are determined by the 4 zeros of \( f''(x) \). It may be easier to look at the factors.

\[ f''(x) = 30(x^2 - 1)(x^2 - \frac{1}{5}) = 30(x + 1)(x - 1)(x + \frac{1}{\sqrt{5}})(x - \frac{1}{\sqrt{5}}) \]

The factors change sign at \( x = -1 \), \( x = -\frac{1}{\sqrt{5}} \), \( x = \frac{1}{\sqrt{5}} \), and \( x = 1 \). If \( x < -1 \), all four factors are negative, and so \( f''(x) \) is positive. Then the sign of \( f''(x) \) alternates.

Conclusion: \( f(x) \) is concave up on the intervals \((−∞, -1)\), \((-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}})\), and \((1, +∞)\)

\( f(x) \) is concave down on the intervals \((-1, -\frac{1}{\sqrt{5}})\) and \((\frac{1}{\sqrt{5}}, 1)\)

10. (12 points; p201 Ex 3) A water tank has the shape of an inverted circular cone with a base radius 2m and height 4m. If water is being pumped into the tank at a rate of 2m³/min, find the rate at which the water level is rising when the water is 3m deep. \textit{Hint}: The volume of a cone is \( V = \frac{1}{3} \pi r^2 h \).

Comments (10/26/06): Page numbers have changed in the 5th edition; this problem is now on page 200.

I would say that this is a pretty hard exam. When I gave it in 2000, there were 4 A’s, 7 B’s, 10 C’s, 3 D’s, and 3 F’s. The class average was 75