MATH 229, Section 6 TEST II NAME SOLUTIONS

Prof. J. Beachy 10/27/2000 NO CALCULATORS! Show all necessary work.

1. (20 points) Find the derivative $f'(x)$ or $\dfrac{dy}{dx}$.

(a) (p157 #14) $f(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}} = x^{1/2} - x^{-1/2}$ $f'(x) = (1/2)x^{-1/2} + (1/2)x^{-3/2}$

(b) (p183 #17) $f(x) = (3x-2)^{10}(5x^2 - x + 1)^{12}$ Use the product rule

$f'(x) = (10)(3x-2)^9(3)(5x^2 - x + 1)^{12} + (3x-2)^{10}(12)(5x^2 - x + 1)^{11}(10x - 1)$

(c) (p215 #31) $y = \sin(\tan(\sqrt{1+x^3}))$ Use the chain rule (3 times)

$\dfrac{dy}{dx} = \cos(\tan(\sqrt{1+x^3})) \cdot (\sec^2(\sqrt{1+x^3})) \cdot (1/2)(1+x^3)^{-1/2}(3x^2)$

(d) (p187 Ex 2) $x^3 + y^3 = 6xy$ Use implicit differentiation (See the text for the solution)

2. (5 points; p197 #17) Find the second derivative $f''(x)$ for $f(x) = \tan(3x)$. Use the chain rule.

$f'(x) = (\sec^2(3x))(3) = 3(\sec(3x))^2$ $f''(x) = 6(\sec(3x)) \cdot \sec(3x) \cdot \tan(3x) \cdot (3)$

3. (6 points; p176 #40) $\displaystyle\lim_{x\to 0} \frac{\tan x}{4x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{1}{4\cos x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{4\cos x} = 1 \cdot \frac{1}{4\cos 0} = \frac{1}{4}$

4. (7 points; p157 #64) Find the equations of the tangent lines to the curve $y = \dfrac{x-1}{x+1}$ that are parallel to the line $x - 2y = 2$. $x - 2y = 2$ $2y = x - 2$ $y = \frac{1}{2}x - 1$

We need to solve $y' = \frac{1}{2}$. Using the quotient rule, we get $y' = \dfrac{1(x+1) - (x-1)(1)}{(x+1)^2} = \dfrac{x+1-x+1}{(x+1)^2} = \dfrac{2}{(x+1)^2}$. Setting $\dfrac{2}{(x+1)^2} = \dfrac{1}{2}$, we get $4 = (x+1)^2$, so $x + 1 = \pm 2$, or $x = 1, -3$. When $x = 1$, $y = 0$, and the corresponding tangent line is $y = \frac{1}{2}(x-1)$. When $x = -3$, $y = 2$, and the corresponding tangent line is $y = \frac{1}{2}(x+3) + 2$.

5. (6 points; p211 #24) For $y = \sqrt{1-x}$, find the differential dy and evaluate dy for $x = 0$ and $dx = 0.02$.

$\dfrac{dy}{dx} = \frac{1}{2}(1-x)^{-1/2} = \dfrac{1}{2\sqrt{1-x}}$. When $x = 0$, $\dfrac{dy}{dx} = \dfrac{1}{2}$, and so $dy = \dfrac{dy}{dx}dx = \frac{1}{2} \cdot 0.02 = 0.01$.

6. (6 points; p217 #79) A window has the shape of a square surmounted by a semicircle. The base of the window is measured as having width $60cm$, with a possible error of $0.1cm$. Use differentials to estimate the maximum error possible in computing the area of the window.

Let x be the width of the window. Then the area of the window is $A(x) = x^2 + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = x^2 + \frac{1}{8}\pi x^2$, and $\Delta A \simeq A'(x)\Delta x = (60^2 + \frac{1}{8}\pi(60)^2)(0.1) = (3600 + \pi(15)(30))(0.1) = (3600 + 4500\pi)(0.1) = 360 + 450\pi$.

7. (8 pts; p167 #8) If a ball is thrown vertically upward with a velocity of 80 ft/sec, then its height after t seconds is $s = 80t - 16t^2$.

(a) What is the maximum height reached by the ball?

We have $s'(t) = 80 - 32t$. Set $s'(t) = 0$, so $80 - 32t = 0$, or $t = 2.5$. Then the corresponding height is $s = 80(2.5) - 16(6.25) = 200 + 100 = 300$ft.

(b) What is the velocity of the ball when it is 96 ft above the ground on the way up?

When $s = 96$, we get $96 = 80t - 16t^2$ or $16t^2 - 80t + 96 = 0$. Dividing by 16 gives $t^2 - 5t + 6 = 0$, so $(t-2)(t-3) = 0$. We get $t = 2$ or $t = 3$. The first solution must correspond to a height of 96 ft on the way up. Substituting $t = 2$ into the formula for the derivative gives $s'(2) = 80 - 64 = 16$, so the velocity is 16ft/sec.

8. (17 pts; p248 #11) For $f(x) = x^3 - 12x + 1$, graph the function after finding
(a) the intervals on which f is increasing or decreasing;

We need to analyse the sign of $f'(x) = 3x^2 - 12$. We have $f'(x) = 3(x^2 - 4) = 3(x+2)(x-2)$, so $f'(x) = 0$ when $x = -2$ or $x = 2$. We need to look at the sign of $f'(x)$ on the intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, +\infty)$. Test at $x = -3$, $x = 0$, and $x = 3$. We get $f'(-3) = (3)(-1)(-5)$, which is positive, $f'(0) = (3)(2)(-2)$, which is negative, and $f'(3) = (3)(5)(1)$, which is positive.

Conclusion: $f(x)$ is increasing on $(-\infty, -2)$ and $(2, +\infty)$ and decreasing on $(-2, 2)$.

(b) the local maximum and minimum values of f;

Using the first derivative test we can see that there is a local maximum at $x = -2$ and a local minimum at $x = 2$. The corresponding points on the curve are $(-2, 17)$ and $(2, -15$, since $f(-2) = -8 + 24 + 1 = 17$ and $f(2) = 8 - 24 + 1 = -15$.

(c) the intervals of concavity and the inflection points.

We need to analyse the sign of $f''(x) = 6x$. The graph is concave down on $(-\infty, 0)$, and concave up on $(0, +\infty)$, so $(0, 1)$ is the inflection point.

9. (13 points; p248 #32) For the function $f(x) = (x^2 - 1)^3$, find
(a) the intervals on which f is increasing or decreasing;

We have $f'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2$. Setting $f'(x) = 0$ we get $6x(x^2 - 1)^2 = 0$, so either $x = 0$ or $x^2 - 1 = 0$, giving $x = \pm 1$. We need to test the sign of $f'(x)$ on the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

Instead of choosing a point in each interval, it is probably better to analyse the sign of each factor of $f'(x)$. Since $f'(x) = 6x(x^2 - 1)^2$, look at the factors $6x$ and $(x^2 - 1)^2$ separately. The first factor is negative for $x < 0$ and positive for $x > 0$. The second factor is never negative. Conclusion: $f'(x) \leq 0$ for $x < 0$ and $f'(x) \leq 0$ for $x > 0$, so $f(x)$ is increasing on $(-\infty, -1)$ and $(-1, 0)$ and decreasing on the intervals $(0, 1)$ and $(1, \infty)$.

(b) the local maximum and minimum values of f;

Although $f'(x) = 0$ for $x = -1, 0, 1$, the derivative does not change sign at $x = -1$ and $x = 1$, so these points are neither a relative max nor a relative min. At $x = 0$, the function changes from decreasing to increasing, so there is a relative minimum at $(0, -1)$. (Note that $f(0) = (0^2 - 1)^3 = -1$.

(c) the intervals of concavity and the inflection points.

Since $f'(x) = 6x(x^2 - 1)^2$, we need to use the product rule to find $f''(x)$. We get
$f''(x) = 6(x^2 - 1)^2 + (6x)(2)(x^2 - 1)(2x) = (x^2 - 1)(6x^2 - 6 + 24x^2) = (x^2 - 1)(30x^2 - 6) = 6(x^2 - 1)(5x^2 - 1)$.

Again, we could test values in each of the 5 intervals that are determined by the 4 zeros of $f''(x)$. It may be easier to look at the factors.

$f''(x) = 30(x^2 - 1)(x^2 - \frac{1}{5}) = 30(x + 1)(x - 1)(x + \frac{1}{\sqrt{5}})(x - \frac{1}{\sqrt{5}}) = 30(x + 1)(x + \frac{1}{\sqrt{5}})(x - \frac{1}{\sqrt{5}})(x - 1)$

The factors change sign at $x = -1$, $x = -\frac{1}{\sqrt{5}}$, $x = \frac{1}{\sqrt{5}}$, and $x = 1$. If $x < -1$, all four factors are negative, and so $f''(x)$ is positive. Then the sign of $f''(x)$ alternates.

Conclusion: $f(x)$ is concave up on the intervals $(-\infty, -1)$, $(-\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}})$, and $(1, \infty)$

$f(x)$ is concave down on the intervals $(-1, -\frac{1}{\sqrt{5}})$ and $(\frac{1}{\sqrt{5}}, 1)$

10. (12 points; p201 Ex 3) A water tank has the shape of an inverted circular cone with a base radius $2m$ and height $4m$. If water is being pumped into the tank at a rate of $2m^3$/min, find the rate at which the water level is rising when the water is $3m$ deep. *Hint:* The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.

Comments (10/26/06): Page numbers have changed in the 5th edition; this problem is now on page 200.

I would say that this is a pretty hard exam. When I gave it in 2000, there were 4 A's, 7 B's, 10 C's, 3 D's, and 3 F's. The class average was 75