1. [5 points; p136] State the limit definition of the derivative of a function.

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$, if the limit exists

2. [15 pts] Find these derivatives. You do not need to simplify your answers.

(a) [p156#13,16] $f(x) = x^2 + \frac{1}{x^2} + x^{4/3} - x^{2/3} = x^2 + x^{-2} + x^{4/3} - x^{2/3}$

$f'(x) = 2x + -2x^{-3} + \frac{4}{3}x^{1/3} - \frac{2}{3}x^{-1/3}$

(b) [p156#21] $f(x) = (x^3 - x + 1)(x^{-2} + 2x^{-3})$ *Use the product rule.*

$f'(x) = [\frac{d}{dx}(x^3 - x + 1)](x^{-2} + 2x^{-3}) + (x^3 - x + 1)[\frac{d}{dx}(x^{-2} + 2x^{-3})]$

$= (3x^2 - 1)(x^{-2} + 2x^{-3}) + (x^3 - x + 1)(-2x^{-3} - 6x^{-4})$

(c) [p156#26] $f(x) = \frac{\sqrt{x} - 1}{\sqrt{x} + 1} = \frac{x^{1/2} - 1}{x^{1/2} + 1}$ *Use the quotient rule.*

$$f'(x) = \frac{[\frac{d}{dx}(x^{1/2} - 1)](x^{1/2} + 1) - (x^{1/2} - 1)[\frac{d}{dx}(x^{1/2} + 1)]}{(\sqrt{x} + 1)^2}$$

$$= \frac{(\frac{1}{2}x^{-1/2})(x^{1/2} + 1) - (x^{1/2} - 1)(\frac{1}{2}x^{-1/2})}{(\sqrt{x} + 1)^2} = \frac{x^{-1/2}}{(\sqrt{x} + 1)^2} = \frac{1}{\sqrt{x}(\sqrt{x} + 1)^2}$$

3. [5 pts; p48#38] Find (and simplify) the composite function $f(f(x))$, for $f(x) = \frac{x - 1}{x + 1}$.

$$f(f(x)) = \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1} = \frac{\left(\frac{x-1}{x+1} - 1\right)(x + 1)}{\left(\frac{x-1}{x+1} + 1\right)(x + 1)} = \frac{(x - 1) - (x + 1)}{(x - 1) + (x + 1)} = \frac{-2}{2x} = -\frac{1}{x}$$

4. [25 pts] Compute these limits:

(a) [p92#20] $\lim_{x\to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x\to 2} \frac{(x - 2)(x + 3)}{(x - 2)(x + 2)} = \lim_{x\to 2} \frac{x + 3}{x + 2} = \frac{2 + 3}{2 + 2} = \frac{5}{4}$

Comment: Substituting $x = 2$ gives you the form $\frac{0}{0}$, which guarantees that $x - 2$ is a factor of both the numerator and denominator.

(b) [p83#22] $\lim_{x\to 5^-} \frac{6}{x - 5} = -\infty$

The graph of $y = \frac{6}{x - 5}$ has a vertical asymptote at $x = 5$, and as you approach from the left, with $x < 5$, you can see that the fractions are negative and get larger and larger in absolute value.

Comment: Substituting $x = 5$ gives you the form $\frac{6}{0}$, which is the clue that the limit should be infinite. You may find it easiest to just graph the function.

(c) [p92#38] $\lim_{x\to -4^+} \frac{|x + 4|}{x + 4} = \lim_{x\to -4^+} \frac{x + 4}{x + 4} = 1$

Comment: Since $x \to -4^+$, we need to consider values of x close to -4, with $x > -4$. For these values of x we get $|x + 4| = x + 4$, so we can make this substitution.

(d) [p92#16] $\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{x\to 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)} = \lim_{x\to 1} \frac{x^2 + x + 1}{x + 1} = \frac{3}{2}$

(e) [p92#25] $\lim_{x\to 0} \left[\frac{1}{x\sqrt{1 + x}} - \frac{1}{x}\right] = \lim_{x\to 0} \left[\frac{1}{x\sqrt{1 + x}} - \frac{\sqrt{1 + x}}{x\sqrt{1 + x}}\right] = \lim_{x\to 0} \left[\frac{1 - \sqrt{1 + x}}{x\sqrt{1 + x}}\right]$

$$= \lim_{x\to 0} \left[\frac{(1 - \sqrt{1 + x})}{(x\sqrt{1 + x})}\right] \left[\frac{(1 + \sqrt{1 + x})}{(1 + \sqrt{1 + x})}\right] = \lim_{x\to 0} \frac{1 - (1 + x)}{(x\sqrt{1 + x})(1 + \sqrt{1 + x})} = \lim_{x\to 0} \frac{-x}{x\sqrt{1 + x}(1 + \sqrt{1 + x})}$$

$$= \lim_{x\to 0} \frac{-1}{\sqrt{1 + x}(1 + \sqrt{1 + x})} = \frac{-1}{\sqrt{1 + 0}(1 + \sqrt{1 + 0})} = -\frac{1}{2}$$

5. [8 pts] *Use the limit definition of the derivative* to find $f'(x)$, for the function $f(x) = x^2$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} 2x + h = 2x$$

6. [10 pts; p125#23] For the function $f(x)$ given below, check whether or not $f(x)$ is continuous at $x = 0$ and at $x = 3$. Explain your answer by computing the value of the function, the limit from the left, and the limit from the right.

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \le x < 3 \\ (x-3)^2 & \text{if } 3 \le x \end{cases}$$

$f(0) = 3 - 0 = 3 \quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sqrt{-x} = \sqrt{0} = 0 \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (3-x) = 3 - 0 = 3$

Since these values aren't equal, the function is discontinuous at $x = 0$.

$f(3) = (3-3)^2 = 0 \quad \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 3 - x = 3 - 3 = 0 \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x-3)^2 = (3-3)^2 = 0$

Since these values are equal, the function is continuous at $x = 3$.

7. [7 pts; p47#18] On the axes below, graph the function $y = \dfrac{2x+3}{x+1}$.

Substituting $x = -1$ gives $\frac{1}{0}$, an indication that there will be a vertical asymptote. There is a horizontal asymptote at $y = 2$, which you can see by computing $\displaystyle\lim_{x\to+\infty} \frac{2x+3}{x+1} = \lim_{x\to+\infty} \frac{2+3/x}{1+1/x} = 2$.

If you use long division of polynomials to simplify the formula for y, you can see that $\dfrac{2x+3}{x+1} = 2 + \frac{1}{x+1}$. You should then recognize that you can graph $y = \dfrac{2x+3}{x+1}$ by shifting the graph of $y = \dfrac{1}{x}$ one unit to the left, and up two units. You should compute several pairs of coordinates and plot them carefully. You should certainly include $(0, 3)$ and $(-2, 1)$.

8. [9 pts; 3.3 Ex4] Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

The tangent line is horizontal when the derivative is zero, so you need to solve the equation $4x^3 - 12x = 0$. You can factor to get $4x(x^2 - 3)$ so $x = 0$, $x = \sqrt{3}$, or $x = -\sqrt{3}$. The corresponding y-coordinates on the graph are 4 and -5, so the answer is that the tangent line is horizontal at the points $(0, 4)$, $(\sqrt{3}, -5)$, and $(-\sqrt{3}, -5)$.

9. [6 points; p114#43] Use the Intermediate Value Theorem to explain why the equation $x^3 - 3x + 1 = 0$ has a solution. Find an interval that contains this solution.

Try some integer values in the polynomial: $f(0) = 1$, $f(1) = 1^3 - 3 + 1 = -1$, $f(2) = 2^3 - 3(2) + 1 = 3$. The Intermediate Value Theorem guarantees a root between $x = 0$ and $x = 1$, since $f(x)$ changes sign on this interval. You could also give the interval $[1, 2]$, since $f(x)$ also changes sign on this interval. In fact, since $f(-2) = (-2)^3 - 3(-2) + 1 = -8 + 6 + 1 = -1$, and $f(-1) = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3$, there is a third root in the interval $[-2, -1]$. Giving any one of these intervals will answer the question.

10. [10 pts; 3.2 Ex4] *Use the limit definition of the derivative* to find $f'(x)$, for the function $f(x) = \sqrt{x-1}$. Check your answer by using the derivative formulas.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)-1} - \sqrt{x-1}}{h}$$

$$= \lim_{h\to 0} \left[\frac{\sqrt{(x+h)-1} - \sqrt{x-1}}{h}\right] \left[\frac{\sqrt{(x+h)-1} + \sqrt{x-1}}{\sqrt{(x+h)-1} + \sqrt{x-1}}\right] = \lim_{h\to 0} \frac{(x+h) - 1 - (x-1)}{h(\sqrt{(x+h)-1} + \sqrt{x-1})}$$

$$= \lim_{h\to 0} \frac{x + h - 1 - x + 1}{h(\sqrt{(x+h)-1} + \sqrt{x-1})} = \lim_{h\to 0} \frac{h}{h(\sqrt{(x+h)-1} + \sqrt{x-1})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{(x+h)-1} + \sqrt{x-1}} = \frac{1}{\sqrt{x-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$