1. (10 points) (a) State the definition of the derivative of a function.

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, if the limit exists

(b) State the definition of the limit of a function.

$\lim_{x \to a} f(x0 = L$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$

2. (15 points; p134 #9) (a) Find the derivative of the function $f(x) = x^3 - 5x + 1$, *using the limit definition of the derivative.*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - 5(x+h) + 1 - (x^3 - 5x + 1)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - 5(x+h) + 1 - (x^3 - 5x + 1)}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 5x - 5h + 1 - x^3 + 5x - 1}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - 5h}{h} = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - 5)}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 - 5 = 3x^2 - 5$$

(b) Find the equation of the tangent line to the curve $y = x^3 - 5x + 1$ at $x = 1$.

The slope of the tangent line is $f'(1) = 3(1)^2 - 5 = 3 - 5 = 2$. The tangent line goes through $(1, f(1)) = (1, -3)$ since $f(1) = 1^3 - 5(1) + 1 = 2 - 5 = -3$. The equation of the line through (x_0, y_0) with slope m is given by $y = m(x - x_0) + y_0$, so the equation of the tangent line is $y = -2(x - 1) - 3$.

(c) Find the equation of the tangent line to the curve $y = x^3 - 5x + 1$ at $x = 0$.

This time the point on the curve is $(1, 0)$, and the slope is $f'(0) = 3(0)^2 - 5 = -5$, so the tangent line is $y = -5(x - 0) + 1 = -5x + 1$.

3. (25 points; p125 #8, #10, #13; p92 #16, #21) Compute the following limits:

(a) $\lim_{x \to -1} \dfrac{x^2 - x - 2}{x^2 + 3x + 2} = \lim_{x \to -1} \dfrac{(x+1)(x-2)}{(x+1)(x+2)} = \lim_{x \to -1} \dfrac{x-2}{x+2} = \dfrac{-3}{1} = -3$

Comment: First substitute $x = -1$. You get the form $\dfrac{0}{0}$, which guarantees that $x + 1$ is a factor of both the numerator and denominator.

(b) $\lim_{x \to -6^+} \dfrac{x}{x+6} = -\infty$

The graph has a vertical asymptote at $x = -6$, and as you approach from the right, with $x > -6$, you the fractions are negative and get larger and larger in absolute value.

Comment: Substituting $x = -6$ gives you the form $\dfrac{-6}{0}$, which is the clue that the limit should be infinite. You may find it easiest to just graph the function.

(c) $\lim_{x \to 8^-} \dfrac{|x - 8|}{x - 8} = \lim_{x \to 8^-} \dfrac{-(x-8)}{x-8} = \lim_{x \to 8^-} -1 = -1$

Comment: Since $x \to 8^-$, we need to consider values of x close to 8, with $x < 8$. For these values of x we must have $|x - 8| = -(x - 8)$, so we can make this substitution.

(d) $\lim_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1} = \lim_{x \to 1} \dfrac{(x-1)(x^2 + x + 1)}{(x-1)(x+1)} = \lim_{x \to 1} \dfrac{x^2 + x + 1}{x + 1} = \dfrac{3}{2}$

(e) $\lim_{t \to 0} \dfrac{\sqrt{2-t} - \sqrt{2}}{t} = \lim_{t \to 0} \dfrac{(\sqrt{2-t} - \sqrt{2})(\sqrt{2-t} + \sqrt{2})}{t(\sqrt{2-t} + \sqrt{2})} = \lim_{t \to 0} \dfrac{2 - t - 2}{t(\sqrt{2-t} + \sqrt{2})}$

$= \lim_{t \to 0} \dfrac{-t}{t(\sqrt{2-t} + \sqrt{2})} = \lim_{t \to 0} \dfrac{-1}{\sqrt{2-t} + \sqrt{2}} = \dfrac{-1}{2\sqrt{2}}$

4. (8 pts; p48 #38) Given $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{x-1}{x+1}$, find and simplify the formulas for $f(g(x))$ and $g(f(x))$.

$$f(g(x)) = \frac{1}{\frac{x-1}{x+1} - 1} = \frac{x+1}{(x-1) - (x+1)} = \frac{x+1}{-2} = \frac{-1}{2}(x+1)$$

$$g(f(x)) = \frac{\frac{1}{x-1} - 1}{\frac{1}{x-1} + 1} = \frac{1 - 1(x-1)}{1 + 1(x-1)} = \frac{1 - x + 1}{1 + x - 1} = \frac{2 - x}{x}$$

5. (10 pts; p113 #34) For the function $f(x)$ given below, check whether or not $f(x)$ is continuous at $x = -1$ and at $x = 1$. Explain your answer by computing the value of the function, the limit from the left, and the limit from the right.

$$f(x) = \begin{cases} 2x+1 & \text{if } x \leq -1 \\ 3x & \text{if } -1 < x < 1 \\ 2x-1 & \text{if } 1 \leq x \end{cases}$$

$f(-1) = 2(-1) + 1 = -1$ $\quad \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} 2x + 1 = 2(-1) + 1 = -1$ $\quad \lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} 3x = 3(-1) = -3$ Since these values aren't equal, the function is discontinuous at $x = -1$

$f(1) = 2(1) - 1 = 1$ $\quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3x = 3(1) = 3$ $\quad \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x - 1 = 2(1) - 1 = 1$
Since these values aren't equal, the function is discontinuous at $x = -1$

Comment: Approaching -1 from the left means $x < -1$, so the first formula $2x+1$ applies. Approaching -1 from the right means $-1 < x$, so the second formula $3x$ applies.

Approaching 1 from the left means $x < 1$, so the second formula $3x$ applies. Approaching 1 from the right means $1 < x$, so the third formula $2x - 1$ applies.

6. (7 pts; p47 #18) On the axes below, graph the function $y = 2 + \dfrac{1}{x+1}$.

The graph of $y = \dfrac{1}{x}$ is shifted 1 unit to the left, and up 2 units. You should draw the asymptotes, and you should compute several pairs of coordinates and plot them carefully. You should certainly include $(0, 3)$ and $(-2, 1)$.

7. (9 points) The derivative of the function $f(x) = x^4 - 4x^3 + 4x^2 + 1$ is $f'(x) = 4x^3 - 12x^2 + 8x$. Use this formula to find the points on the curve $y = x^4 - 4x^3 + 4x^2 + 1$ at which the tangent line to the curve is a horizontal line.

The tangent line is horizontal when $f'(x) = 0$, so you need to solve the equation $4x^3 - 12x^2 + 8x = 0$. You can factor to get $4x(x^2 - 3x + 2) = 4x(x-1)(x-2) = 0$, so $x = 0$, $x = 1$, or $x = 2$. The corresponding y-coordinates on the graph are found by substituting into $f(x)$. Since $f(0) = 1$, $f(1) = (1)^4 - 4(1)^3 + 4(1)^2 + 1 = 1 - 4 + 4 + 1 = 2$, and $f(2) = (2)^4 - 4(2)^3 + 4(2)^2 + 1 = 16 - 32 + 16 + 1 = 1$, the answer is that the tangent line is horizontal at the points $(0, 1)$, $(1, 2)$, and $(2, 1)$.

8. (6 points; p114 #43) Use the Intermediate Value Theorem to prove that the polynomial $f(x) = x^3 - 3x + 1$ has a root. Find an interval that contains this root.

Try some integer values in the polynomial: $f(0) = 1$, $f(1) = 1^3 - 3 + 1 = -1$, $f(2) = 2^3 - 3(2) + 1 = 3$. The Intermediate Value Theorem guarantees a root between $x = 0$ and $x = 1$, since $f(x)$ changes sign on this interval. You could also give the interval $[1, 2]$, since $f(x)$ also changes sign on this interval. In fact, since $f(-2) = (-2)^3 - 3(-2) + 1 = -8 + 6 + 1 = -1$, and $f(-1) = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3$, there is a third root in the interval $[-2, -1]$.

9. (10 points; p145 #24) *Use the limit definition of the derivative* to find $f'(x)$, for the function $f(x) = \dfrac{1}{x^2}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2}{(x+h)^2 x^2} - \frac{(x+h)^2}{(x+h)^2 x^2} \right] = \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2 - (x+h)^2}{(x+h)^2 x^2} \right] = \lim_{h \to 0} \frac{1}{h} \left[\frac{x^2 - x^2 - 2xh - h^2}{(x+h)^2 x^2} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{h(-2x - h)}{(x+h)^2 x^2} \right] = \lim_{h \to 0} \frac{-2x - h}{(x+h)^2 x^2} = \frac{-2x}{x^2 x^2} = \frac{-2}{x^3}$$