The test will cover Sections 6.1–6.4, 7.1, 7.2*–7.4*, 10.4 **No calculators will be allowed.**

§6.1 Area between two curves

The area of a rectangle is (height)×(width). If the top and bottom are defined by curves, we need to replace this formula with (average height)×(width). The integral is designed to do just this.

Type 1. If the top is $y_T = f_2(x)$ and the bottom is $y_B = f_1(x)$, then the area of a thin *vertical* slice is $(y_T - y_B) \cdot \Delta x$. (See Figure 5 on page 376). The total area between the curves is given by

$$\text{Area} = \int_{x=a}^{x=b} [y_T - y_B]\, dx = \int_a^b [f_2(x) - f_1(x)]\, dx\ .$$

Type 2. If the sides are curved instead of the top and bottom, then we need to find the average width, by subtracting the left hand side from the right hand side and letting an integral do the averaging. Suppose that $x_R = g_2(y)$ is the right hand side and $x_L = g_1(y)$ defines the left hand side. Then we would approximate the area by thin *horizontal* slices of area $(x_R - x_L) \cdot \Delta y$. (See Figure 12 on page 379). The total area between the curves is given by

$$\text{Area} = \int_{y=a}^{y=b} [x_R - x_L]\, dy = \int_a^b [g_2(y) - g_1(y)]\, dy\ .$$

For future reference when working with solids of revolution, remember that if you have y as a function of x you need to use *vertical* slices, but if x is a function of y, you need to use *horizontal* slices to approximate the area.

§6.2 General volumes

The volume of a rectangular box or a circular cylinder is given by the formula (area of the base)×(height). This works because the cross section areas are always the same as the area of the base. But if the cross sections change shape as the height increases, then we could find the volume by integrating to get (average cross section area)×(height). To do this, we need a formula $A(y)$ that gives the cross section area at a height of y. This value $A(y)$ represents the area of a *horizontal* slice taken at a height of y. For the volume we then have these formulas.

$$\text{Volume} = \int_{y=a}^{y=b} A(y)\, dy \qquad \text{or} \qquad \text{Volume} = \int_{x=a}^{x=b} A(x)\, dx$$

We get the second formula if the cross sections represent *vertical* slices instead of *horizontal* slices. Then we integrate the vertical cross sections $A(x)$ with respect to x.

§6.2, 6.3 Solids of revolution via washers and shells

In these sections, a region in the plane is rotated around an axis to produce a "solid of revolution". These regions are described in exactly the same way as when finding their area in Section 6.1, so that is a starting point. We again look at two cases: (Type 1) the top and bottom are described using $y_T = f_2(x)$ and $y_B = f_1(x)$, or (Type 2) the right and left hand side are described using $x_R = g_2(x)$ and $x_L = g_1(x)$.

Type 1. The top and bottom of the region are described by $y_T = f_2(x)$ and $y_B = f_1(x)$, and the area is approximated by thin *vertical* slices. If you rotate a thin vertical slice about a *horizontal* axis $y = L$, (perpendicular to the slice) you get a "washer". The formula to use for "washers" is

$$\text{Volume} = \int_{x=a}^{x=b} \left[\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\right]\, dx = \int_a^b \left[\pi(f_2(x) - L)^2 - \pi(f_1(x) - L)^2\right]\, dx$$

If you rotate a thin vertical slice about a *vertical* axis $x = L$, (parallel to the slice) you get a "cylindrical shell". The formula to use for "shells" is

$$\text{Volume} = \int_{x=a}^{x=b} 2\pi(\text{radius})(\text{height})\, dx = \int_a^b 2\pi(x - L)(f_2(x) - f_1(x))\, dx$$

Notice that you are integrating with respect to x in both formulas, and the limits have not changed–they are the same limits you would have used to find the area between the two curves.

Type 2. The right side of the region is $x_R = g_2(x)$, the left side is $x_L = g_1(x)$, and we need to integrate with respect to y. The area is approximated by thin *horizontal* slices. If you rotate a thin horizontal slice about a *vertical* axis $x = L$, (perpendicular to the slice) you get a "washer". The formula to use for "washers" is

$$\text{Volume} = \int_{y=a}^{y=b} \left[\pi(\text{outer radius})^2 - \pi(\text{inner radius})^2\right] dy = \int_a^b \left[\pi(g_2(y) - L)^2 - \pi(g_1(y) - L)^2\right] dy$$

If you rotate a thin horizontal slice about a *horizontal* axis $y = L$, (parallel to the slice) you get a "cylindrical shell". The formula to use for "shells" is

$$\text{Volume} = \int_{x=a}^{x=b} 2\pi(\text{radius})(\text{height})\, dy = \int_a^b 2\pi(y - L)(g_2(y) - g_1(y))\, dy$$

Again, the functions and the limits are the same ones you would have used to find the area between the two curves that give the left and right boundaries.

§6.4, Work

If a force is constant, the work it does is defined to be the force times the distance through which it acts. If the force is variable, the we need to use an integral to find the *average* force times the distance. If $f(x)$ gives the force at position x, then the work done by the force is

$$\text{Work} = \int_a^b f(x)\, dx\ .$$

Sometimes it is easiest to find the total work done by breaking the problem into small segments to see how to define the integral (see Examples 4 and 5 on page 400).

§7.1, Inverse functions

The inverse of a function is supposed to "undo" whatever the function does. Examples: the cube root is the inverse of the cube function, so the inverse of $f(x) = x^3$ is $g(x) = \sqrt[3]{x}$. Dividing by 2 is the opposite of multiplying by two, so the inverse of $f(x) = 2x$ is $g(x) = \frac{1}{2}x$.

We can describe this relationship by giving the two formulas $f(g(x)) = x$ and $g(f(x)) = x$. If $f(x) = y$, then $g(y) = x$, so if (x, y) is on the graph of $f(x)$, then (y, x) is on the graph of $g(x)$. This explains why the graphs of $f(x)$ and its inverse $g(x)$ are symmetric about the line $y = x$ (see Figure 9 on page 417). This symmetry explains how the vertical line test for a function becomes the horizontal line test to check for the existence of an inverse (see page 414). To find the formula for the inverse of $y = f(x)$, interchange x and y and solve for y in terms of x. Again, this works because (x, y) is on the graph of $f(x)$ if and only if (y, x) is on the graph of the inverse $f^{-1}(x)$.

§7.2*–7.4*, The natural log and exponential functions

This is one of the most important pairs of inverse functions. We start out by defining $\ln x$ to be the antiderivative of the function $f(x) = \dfrac{1}{x}$ (remember that we can't use the power rule to find and antiderivative for $f(x) = x^{-1}$).

$$\ln x = \int_1^x \frac{1}{t}\, dt \qquad x > 0$$

We then define e^x to be the inverse function for $\ln x$, which gives us these basic formulas.

$$\ln(e^x) = x \quad \text{and} \quad e^{\ln x} = x$$

Because they represent inverse functions, the graphs are symmetric about the line $y = x$. You might want to draw $y = e^x$ first because it may be more familiar (it goes through the points $(0, 1)$ and $(1, e)$, and has $y = 0$ as a horizontal asymptote) and then draw the graph of the natural log (it goes through the points $(1, 0)$ and $(e, 1)$, and has $x = 0$ as a vertical asymptote). See Figure 5 on page 453, Figure 1 on page 460, and Figure 3 on page 462.

Basic facts: $\ln 1 = 0$ $\quad \ln e = 1$ $\quad \ln\left(e^{u(x)}\right) = u(x)$ $\quad e^{u(x)}e^{v(x)} = e^{u(x)+v(x)}$ $\quad \left(e^{u(x)}\right)^k = e^{ku(x)}$

$$\ln u(x)^k = k \ln u(x) \qquad \ln(u(x)v(x)) = \ln u(x) + \ln v(x) \qquad \ln\left(\frac{u(x)}{v(x)}\right) = \ln u(x) - \ln v(x)$$

Since we defined $\ln x$ to be the antiderivative of $\frac{1}{x}$, we have $\frac{d}{dx}\ln x = \frac{1}{x}$. Then from the identity $\ln(e^x) = x$ we can find the derivative of e^x, which is $\frac{d}{dx}e^x = e^x$. The chain rule gives these general formulas.

$$\frac{d}{dx}\ln u(x) = \frac{u'(x)}{u(x)} \qquad \frac{d}{dx}e^{u(x)} = e^{u(x)}\,u'(x) \qquad \int \frac{1}{u}\,du = \ln|u| + C \qquad \int e^u\,du = e^u + C$$

Finally, we need to be able to use bases other than e. The formulas become more complicated (that's the reason for using base e in calculus), but we still have the basic identity that says the functions are inverse functions.

$$a^{\log_a x} = x \quad \text{and} \quad \log_a(a^x) = x$$

To simplify $a^{u(x)}$, use the identity $a = e^{\ln a}$ to get $a^{u(x)} = e^{(\ln a)\cdot u(x)}$. To simplify $\log_a(x)$, take the natural log of $a^{\log_a x} = x$ to get $(\log_a x)(\ln a) = \ln x$, so that we have $\log_a(x) = \frac{\ln x}{\ln a}$. I would recommend always converting to base e, but if you want to memorize more formulas, here are the general formulas.

$$\frac{d}{dx}a^{u(x)} = e^{u(x)}(\ln a)u'(x) \quad \text{and} \quad \frac{d}{dx}\log_a u(x) = \frac{u'(x)}{(\ln a)u(x)}$$

The derivative of $f(x) = a^x$ is $f'(x) = ka^x$, where $k = f'(0)$. If we choose the base e, then $f(x) = e^x$ has derivative $f'(x) = e^x$, because $f'(0) = 1$. That is the reason for using $e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2.7182818.....$

§10.4, Exponential growth and decay

Applications: The model for "uninhibited growth" (see page 647) is based on the assumption that the rate of growth is proportional to the amount, at any time. That is, $\frac{dP}{dt} = kP(t)$, at time t. The solution to this "differential equation" is

$$P(t) = P_0 e^{kt}\ ,$$

where P_0 is the initial amount (when $t = 0$). The problems usually have enough information to find P_0 and k, and then you can answer additional questions about the situation. The same equation is used for radioactive decay (in this case k is negative because the amount is decreasing). Sometimes the half-life is given, and this can be used to find the decay constant k.