The test will cover Sections 7.5, 7.7, 8.1–8.5, 8.8 **No calculators will be allowed.**

## §7.5 Inverse trig functions

In order for a function to have an inverse function, it's graph must pass the horizontal line test. For the trig functions, this means that we have to restrict the domain.

| | |
|---|---|
| $\sin x$ has domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and range $[-1, 1]$ | $\sin^{-1} x$ has domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ |
| $\cos x$ has domain $[0, \pi]$ and range $[-1, 1]$ | $\sin^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$ |
| $\tan x$ has domain $(-\frac{\pi}{2}, \frac{\pi}{2})$ and range $(-\infty, \infty)$ | $\tan^{-1} x$ has domain $(-\infty, \infty)$ and range $(-\frac{\pi}{2}, \frac{\pi}{2})$<br>$\tan^{-1} x$ has horizontal asymptotes at $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ |

You need to remember the formulas for the derivatives of $\sin^{-1} x$, $\cos^{-1} x$, and $\tan^{-1} x$. Of course, these also give you integration formulas. The differentiation formulas can all be checked by using the basic identities which show that the pairs of functions are inverses. (Note: you can use some of the integration formulas on the reference sheet as differentiation formulas.)

(page 481) $\quad \frac{d}{dx} \sin^{-1} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \cos^{-1} = \frac{-1}{\sqrt{1-x^2}} \qquad \frac{d}{dx} \tan^{-1} = \frac{1}{1+x^2}$

## §7.7 L'Hospital's rule

(page 494) **L'Hospital's Rule:** If $f(x)$ and $g(x)$ have derivatives, and the limit of their quotient has an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit of the derivatives exists (or is $\pm\infty$).

To apply L'Hospitals rule to a product or difference of functions you must rewrite the product or difference as a quotient (see pages 497–498). By taking the natural log you can also handle indeterminate forms of these types: $0^0$, $\infty^0$, or $1^\infty$. You will need to use this approach to find some of the limits used to check convergence of infinite series.

## §8.1 Integration by parts

We have used the reverse of several differentiation formulas in computing integrals. For example, making a substitution uses the reverse of the chain rule. So far we haven't used the product rule. It turns out to be difficult to use in reverse, and is best used as a kind of reduction formula, to turn a complicated integral into an easier one.

(page 512) **Integration by parts:** $\int u dv = uv - \int v du.$

Remember that the goal is to choose $dv$ so that it has a reasonable antiderivative, and $u$ so that its derivative is simpler. If the integral is a product of unlike functions, that is a clue that integration by parts may work. Sometimes we need to simply choose $dv = dx$.

## §8.2 Trig integrals

Strategies for integrating combinations of trig functions are given in the text on pages 520, 522, and 523. The approach is to choose a substitution $u$ by looking for a convenient candidate for $du$. That may be $du = \sin x\, dx$ or $du = \cos x\, dx$, if the integral involves powers of $\sin x$ and $\cos x$. It the integral involves $\tan x$ and $\sec x$, then either $du = \sec^2 x\, dx$ or $du = \sec x \tan x\, dx$.

More specifically, for $\int \sin^m x \cos^n x\, dx$, if one factor occurs to an odd power, use it as $du$, and if both powers are even, use the half-angle identities. For $\int \tan^m x \sec^n x\, dx$, if $n$ is even, let $du = \sec^2 dx$, and if $m$ is odd, let $du = \sec x \tan x dx$. For $\int \sin mx \cos nx\, dx$, or similar integrals, use the trig identities for $\sin\theta \cos\phi$, etc.

If you can't see how to approach the problem, it may work to rewrite everything in terms of $\sin x$ and $\cos x$.

## §8.3 Trig substitutions

This section causes lots of problems for students, but I think that if you start with a basic triangle you can make sense of the different substitutions. If you see a triangle with sides $a$ and $b$, then you will immediately write $c = \sqrt{a^2 + b^2}$ for the hypotenuse, since $a^2 + b^2 = c^2$. Notice that you would also have $b = \sqrt{c^2 - a^2}$.

I've list the cases given on page 526 of the text. Instead of using the recommended substitution in all of the terms, I like to identify them as parts of the original triangle, and then express them as trig functions of $\theta$.

Case 2. $\sqrt{a^2 + x^2}$ This term has to be on the hypotenuse of the "substitution triangle", which means that the sides of the triangle are $x$ and $a$. If you let $\tan\theta = \dfrac{x}{a}$, you get the recommended substitution $x = a\tan\theta$.

Case 1. $\sqrt{a^2 - x^2}$ This means that $a$ has to be on the hypotenuse, and $x$ is one of the sides of the triangle, while $\sqrt{a^2 - x^2}$ is the other side. If you let $\sin\theta = \dfrac{x}{a}$, you get the recommended substitution $x = a\sin\theta$.

Case 3. $\sqrt{x^2 - a^2}$ In this case, $x$ has to be on the hypotenuse, and $a$ must be one of the sides of the triangle, while $\sqrt{x^2 - a^2}$ is the other side. If you let $\sec\theta = \dfrac{x}{a}$, you get the recommended substitution $x = a\sec\theta$.

## §8.4 Partial fractions

To add two rational functions, we find a common denominator and then add the numerators. Sometimes we face an integral that would be much easier to find if we could reverse this process. This idea leads to the technique of partial fractions: break a rational function up into simpler pieces.

The first step is to make sure that the degree of the numerator is less than the degree of the denominator. If it isn't, use long division. Then factor the denominator completely, which means that it will be a product of linear and quadratic factors, possibly to various powers.

After you break the rational function apart, any linear factors have only a constant in the numerator, while quadratic factors have a linear term in the numerator. If a factor is raised to a power, then you must include *all* powers up to that exponent.

Example:
$$\int \frac{x-5}{(x-3)^3(x^2+1)^2}\,dx = \int \left( \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2} \right) dx$$

## §8.5 Putting it all together

**Simplify**. Try to reduce a problem to one which can be done easily or can be solved by a known formula.

- Easy algebraic manipulations: break integrals into sums, exploit identities, reduce rational functions to proper functions by division.
- Obvious substitutions: "inside" functions or "nasty" terms or denominators.

**Classify**. If there is no obvious solution, a systematic approach must be used. The choice of technique is based on the form of the integral.

- Rational functions: factor the denominator and write the function as a sum of simpler functions.
- Products: consider integration by parts, especially if the function is a product of dissimilar functions. In using $\int u\,dv = uv - \int v\,du$, choose $dv$ so that it can be integrated, while $\int v\,du$ is simpler or easier than $\int u\,dv$.
- Trig functions: use "twin pairs" to find a substitution, e.g. $\int f(\sin(x))\cos(x)\,dx$, etc. Reduce powers by using half angle formulas or integration by parts. If all else fails, try $u = \tan(\theta/2)$.
- Special functions: use trig substitutions for $(\sqrt{a^2 \pm u^2})^n$ or $(\sqrt{u^2 - a^2})^n$. For functions of $e^x$, use $u = e^x$. For $\sqrt[n]{ax+b}$, use $u^n = ax + b$.

**Modify**. Problems may have an unfamiliar form that can be modified to put it in a more familiar or simpler form.

- Look for resemblances between the problem and more familiar ones.
- Look to see what additional terms might help to solve a problem. Add the term and then try to compensate for it.
- Some special tricks: rationalize the denominator, or multiply by a "conjugate", e.g. $(1 - \cos x)(1 + \cos x) = \sin^2 x$. Substitute $x = u^n$ to get rid of roots.

## §8.8 Improper integrals

Without some new technique we can't find the area under a curve if the region has infinite length, or if the function has a vertical asymptote. We can only make sense out of these situations if we calculate these areas as a limit of finite areas.

(page 567) **Definition:** We make the following definition for an *improper* integral:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

*provided* the limit is finite *and* $\int_a^t f(x)\,dx$ exists for all $t \geq a$. The improper integral $\int_a^\infty f(x)\,dx$ is called *convergent* if the limit exists and is finite, and *divergent* if the limit does not exist or is infinite. There are similar definitions for $\int_{-\infty}^b f(x)\,dx$ and $\int_{-\infty}^\infty f(x)\,dx$ (also on page 567).

(page 570) **Definition:** If $y = f(x)$ is continuous on $[a, b)$ and has a vertical asymptote at $x = b$, then we make the following definition:

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

provided this limit exists and is finite. The improper integral $\int_a^b f(x)\,dx$ is called *convergent* if the limit exists and is finite, and *divergent* if the limit does not exist or is infinite. We can give similar definitions if $y = f(x)$ has a vertical asymptote at $x = a$ instead of $x = b$, or if the vertical asymptote occurs for $x = c$, where $a < c < b$. *From now on, you must look at any integral* $\int_a^b f(x)\,dx$ *very carefully, to make sure that there is no vertical asymptote in the interval* $[a, b]$ *over which you are integrating.*

(page 572) **Comparison theorem:** Suppose that $f(x) \geq g(x) \geq 0$ on the interval $[a, \infty)$.
(a) If the larger integral $\int_a^\infty f(x)\,dx$ is convergent, then so is the smaller one $\int_a^\infty g(x)\,dx$.
(b) If the smaller integral $\int_a^\infty g(x)\,dx$ is divergent, then so is the larger one $\int_a^\infty f(x)\,dx$.

## §9.1 Arc length

Think about a car traveling along a curving road. Let's suppose that the $x$ and $y$ coordinates of the car are given as a function of time. (This point of view is called a parametric curve, introduced in Section 11.2.) Since the car may not be traveling at a constant speed, we can't use the simple formula "rate times time equals distance". The distance the car travels *can* given by an integral, which should give us the average rate times time. The distance should be the integral of the speed, as a function of time.

The velocity of the car has an $x$-component and a $y$-component, given by $\frac{dx}{dt}$ and $\frac{dy}{dt}$, respectively. The speed at time $t$ is given by $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Taking the integral gives us the arclength (see page 699 for this formula):

$$L = \int_{t=a}^{t=b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt\ .$$

In Section 9.1 the author only deals with the simplified cases in which either $x = t$ or $y = t$. This gives the two formulas on page 585. If $\frac{dy}{dx}$ is a continuous function of $x$ on $[a, b]$, we get the first formula, and if $\frac{dx}{dy}$ is a continuous function of $y$ on $[c, d]$, we get the second formula.

$$L = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx \qquad \text{or} \qquad L = \int_{y=c}^{y=d} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\; dy\ .$$

Integration reference sheet

$$\int \tan u\, du = \ln|\sec u| + C \qquad \int \sec u\, du = \ln|\sec u + \tan u| + C$$

$$\int \cot u\, du = \ln|\sin u| + C \qquad \int \csc u\, du = -\ln|\csc u + \cot u| + C$$

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln|u + \sqrt{u^2 + a^2}| + C$$

$$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln|u + \sqrt{u^2 - a^2}| + C$$

$$\int \frac{du}{u\sqrt{a^2 \pm u^2}} = -\frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 \pm u^2}}{u}\right| + C$$

$$\int \sqrt{a^2 - u^2}du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} + C$$

$$\int \sqrt{u^2 \pm a^2}du = \frac{u}{2}\sqrt{u^2 \pm a^2} \pm \frac{a^2}{2}\ln|u + \sqrt{u^2 \pm a^2}| + C$$

Trig identities

$$\sin(\theta + \phi) = \sin\theta\ \cos\phi + \cos\theta\ \sin\phi \qquad \tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

$$\cos(\theta + \phi) = \cos\theta\ \cos\phi - \sin\theta\ \sin\phi$$

$$\sin\theta\cos\phi = \tfrac{1}{2}(\sin(\theta - \phi) + \sin(\theta + \phi))$$

$$\cos\theta\cos\phi = \tfrac{1}{2}(\cos(\theta - \phi) + \cos(\theta + \phi)) \qquad \cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin\theta\sin\phi = \tfrac{1}{2}(\cos(\theta - \phi) - \cos(\theta + \phi)) \qquad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

For rational functions of $\sin\theta$ and $\cos\theta$, the substitution $u = \tan\frac{\theta}{2}$ gives $d\theta = \frac{2\,du}{1+u^2}$, $\sin\theta = \frac{2u}{1+u^2}$, $\cos\theta = \frac{1-u^2}{1+u^2}$.