Section 12.2

(page 750) **Definition.** The infinite series $\sum_{n=1}^{\infty} a_n$ is called *convergent* if $\lim_{n\to\infty} \sum_{i=1}^{n} a_n$ is a real number s. The real number s is called the *sum* of the series. If the limit is infinite or does not exist, the series is called *divergent*. Thus if the series converges we have a formula like the one for integrals: $\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx$.

$$\sum_{n=1}^{\infty} a_n \quad = \quad \lim_{n\to\infty} \sum_{i=1}^{n} a_n$$

(page 754) **The Divergence Test.** If $\lim_{n\to\infty} a_n \neq 0$, or $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(page 751) A **geometric series** may or may not converge:

$$\sum_{n=1}^{\infty} ar^{n-1} \quad = \quad a + ar + ar^2 + \ldots + ar^n + \ldots \quad = \quad \frac{a}{1-r} \qquad \text{if} \qquad |r| < 1$$

(page 753) The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n} \quad = \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$ is divergent.

(page 755) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent, the so are $\sum_{n=1}^{\infty} ca_n$ and $\sum_{n=1}^{\infty} (a_n \pm b_n)$, and

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n \qquad\qquad \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Section 12.3

(page 760) **The Integral Test.** Assume $f(x)$ is a continuous, positive, decreasing function on $[1, \infty)$ with $a_n = f(n)$.
If $\int_1^{\infty} f(x)\,dx$ converges, then so does $\sum_{n=1}^{\infty} a_n$. If $\int_1^{\infty} f(x)\,dx$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

(page 761) The p**-series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Section 12.4

(page 767) **The Direct Comparison Test.** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with $a_n > 0$ and $b_n > 0$ for all n.

If $\sum_{n=1}^{\infty} b_n$ converges and $a_n \leq b_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} b_n$ diverges and $b_n \leq a_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ diverges.

(page 768) **The Limit Comparison Test.** Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $a_n > 0$ and $b_n > 0$.

If $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and is a positive number, then the two series have exactly the same behavior.

That is, $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Section 12.5

(page 768) **The Alternating Series Test.** The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ (where $b_n > 0$) converges provided (i) $b_n \geq b_{n+1}$ for all n and (ii) $\lim_{n\to\infty} b_n = 0$.

(page 774) *Estimating an alternating series:* If $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges by the alternating series test, then the error $|s - s_n|$ after adding n terms is no larger than the next term b_{n+1}.

Section 12.6

(page 776,777) **Definition.** The series $\sum_{n=1}^{\infty} a_n$ is called *absolutely convergent* if $\sum_{n=1}^{\infty} |a_n|$ is convergent, and *conditionally convergent* if it is convergent but *not* absolutely convergent.

(page 776) **Examples.** The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ is absolutely convergent, while $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ is conditionally convergent.

(page 777) **Theorem.** Any absolutely convergent series is convergent.

(page 778) **The Ratio Test.** Find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$. If the limit is less than 1, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. If the limit is greater than 1 the series is divergent. If the limit equals 1 the test gives no information.

Example. For the geometric series $\sum_{n=0}^{\infty} ar^n$ the Ratio Test gives $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{ar^{n+1}}{ar^n}\right| = \lim_{n\to\infty}|r| = |r|$, and so the result of this test agrees with what we already know.

(page 780) **The Root Test.** Find $\lim_{n\to\infty}\sqrt[n]{|a_n|}$. If the limit is less than 1, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. If the limit is greater than 1 the series is divergent. If the limit equals 1 the test gives no information.

Example. For $\sum_{n=0}^{\infty} ar^n$ the Root Test gives $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{|ar^n|} = \lim_{n\to\infty}\sqrt[n]{|a|}\,\sqrt[n]{|r|^n} = \lim_{n\to\infty} 1\cdot|r| = |r|$, and so the result of this test agrees with what we already know. Note: you can show that $\lim_{n\to\infty}\sqrt[n]{|a|} = 1$ by finding $\lim_{n\to\infty}\ln(|a|^{1/n}) = \lim_{n\to\infty}\frac{1}{n}\ln(|a|)$.

Section 12.7

(page 783) **Strategy for testing convergence**

1,2. Check to see if the series is a geometric series or a p-series.

3. If the series is close to a geometric series or a p-series, then it is probably good to try the Comparison Test or the Limit Comparison Test.

4. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

5. Alternating series fall into their own category. Use the techniques of 12.5 and 12.6.

6. The Ratio Test often works if the series involves factorials or other products.

7. The Root Test often works if $a_n = (b_n)^n$.

8. Unless you see that you can easily integrate, the Integral Test is a last resort.