Section 12.8

(page 785) **Definition.** An infinite series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_2 x^3 + \cdots$ is called a *power series.* This is a special case of a *power series about* a, which has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_2(x-a)^3 + \cdots$$

Example. We can write $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ whenever $|x| < 1$. (See page 790.) Our goal is to understand how to write various functions as power series, and how to use power series to find derivatives and integrals, etc.

(page 787) **Radius of Convergence.** For the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, one of 3 things happens:

(iii) The series converges if $|x-a| < R$ and diverges if $|x-a| > R$, where R is a positive number called the *radius of convergence*;
(ii) The series converges for all x;
(i) The series converges only in the useless case $x = a$.

Note: If the radius of convergence is R, we have to check the two cases $x - a = R$ and $x - a = -R$ separately. After doing this, we can write down the *interval of convergence.* The interval of convergence can be the entire real line, or an interval centered at a. We will need to use all of the convergence tests we have studied.

Section 12.9

(page 792) **Differentiation and Integration.** If $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then the function $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ is a continuous function, and can be differentiated and integrated term by term. These new series both have the same radius of convergence $R > 0$ as the original series.

Section 12.10 Taylor Series

(page 797) **Taylor Series Representation.** If $f(x)$ has a power series representation about a, with $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ for $|x-a| < R$, then the coefficients are given by $c_n = \frac{f^{(n)}(a)}{n!}$. That is:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

This series is called the *Taylor series of the function* $f(x)$ *about* a.

(page 798) **Example.** When $a = 0$, a Taylor series is usually called a *Maclaurin series.* It is easy to calculate the nth derivatives of $f(x) = e^x$, so the Maclaurin series for e^x, which converges for all x, is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(page 799) **Definition.** The polynomial $f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the *nth-degree Taylor polynomial of* $f(x)$ *at* a.

One goal is to approximate functions using Taylor polynomials (see Section 12.12). For example, the first two terms are just the tangent line at a, where we get $f(x) \sim f(a) + f'(a)(x-a)$ when $x \sim a$. The next result gives an estimate on the accuracy of these approximations. If we use $T_n(x)$ for the nth degree Taylor polynomial, and $f(x) = T_n(x) + R_n(x)$, we call $R_n(x)$ the *remainder* of the Taylor series. The next theorem shows that if we can find a maximum M for the $(n+1)$st derivative, then we can estimate the size of the remainder.

(page 799) **Taylor's inequality** Let $R_n(x)$ be the remainder of the Taylor series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.
If $|f^{(n+1)}(x)| \le M$ when $|x-a| \le d$, then

$$|R_n(x)| \le \frac{M}{(n+1)!}\,|x-a|^{n+1} \qquad \text{when } |x-a| \le d\,.$$

Here are some useful series (see page 803). Remember that to construct new series from known ones, you can integrate, differentiate, multiply, or divide.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \sum_{n=0}^{\infty} x^n \qquad (-1,1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty,\infty)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \qquad (-\infty,\infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots \qquad (-\infty,\infty)$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots \qquad [-1,1]$$

Section 12.12

This section uses the nth degree Taylor polynomial for a function $f(x)$ about $x = a$ to approximate the function. Using $x = a$ instead of $x = 0$ is just like finding a tangent line: we need a line through the point $(a, f(a))$ that is as close to the curve as possible. Similarly, we use the nth degree Taylor polynomial to find a polynomial of degree n that goes through the point $(a, f(a))$ and is as close to the graph of the function as possible. Note that we only use information at the point $x = a$, since the coefficients are determined by the various derivatives of the function, all evaluated at $x = a$.