

1. (8 pts each) Find the derivative of each function.

$$(a) y = \ln(\sin x) \quad y' = \frac{\cos x}{\sin x} = \cot x$$

$$(b) y = \ln \left(\sqrt[4]{\frac{x^2-1}{x^2+1}} \right) = \frac{1}{4} \ln(x^2-1) - \frac{1}{4} \ln(x^2+1) \quad y' = \frac{1}{4} \cdot \frac{2x}{x^2-1} - \frac{1}{4} \cdot \frac{2x}{x^2+1} = \frac{x}{x^4-1}$$

$$(c) y = \tan(e^{\sqrt{x}}) \quad y' = \sec^2(e^{\sqrt{x}}) \cdot e^{\sqrt{x}} \cdot \frac{1}{2}x^{-1/2} = \frac{e^{\sqrt{x}}}{2\sqrt{x}} \sec^2(e^{\sqrt{x}})$$

2. (10 pts each) Find each of the following integrals.

$$(a) \int_3^5 \frac{2x}{x^2-5} dx = \int_4^{20} \frac{du}{u} = \ln|u| \Big|_4^{20} = \ln 20 - \ln 4 = \ln 5 = 1.6094, \text{ for } u = x^2 - 5.$$

$$(b) \int_0^2 xe^{4-x^2} dx = -\frac{1}{2} \int_4^0 e^u du = \frac{1}{2} \int_0^4 e^u du = \frac{1}{2} e^u \Big|_0^4 = \frac{1}{2}(e^4 - 1), \text{ for } u = 4 - x^2.$$

$$(c) \int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C, \text{ for } u = \ln x$$

3. (6 pts) Given $f(x) = x^3 - 3$, find the derivative of $f^{-1}(x)$ at 5. Setting $f(x) = 5$ gives $x^3 = 8$, so $f^{-1}(5) = 2$. We have $f'(x) = 3x^2$, and since $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$, the derivative of $f^{-1}(x)$ at 5 is $\frac{1}{3(2)^2} = \frac{1}{12}$.

4. (10 pts) Find the volume of a hemisphere by revolving the following region about the x -axis: the region is in the first quadrant, bounded by the lines $x = 0$, $y = 0$ and the circle $x^2 + y^2 = R^2$.

$$V = \int_{x=0}^{x=R} \pi y^2 dx = \pi \int_0^R (R^2 - x^2) dx = \pi \left(R^2x - \frac{x^3}{3} \right) \Big|_0^R = \pi \left(R^3 - \frac{1}{3}R^3 \right) = \frac{2}{3}\pi R^3$$

5. (20 pts) The region bounded by the curves $y = x^2$ and $x = y^2$ is revolved about the line $x = -1$. Find the volume of the resulting region, using **both** the “washer” method (by slicing) and the method of “cylindrical shells”.

$$\begin{aligned} \text{By washers: } & \pi \int_{x=0}^{x=1} [(\sqrt{y} + 1)^2 - (y^2 + 1)^2] dy = \pi \int_0^1 [y + 2y^{1/2} + 1 - y^4 - 2y^2 - 1] dy \\ & = \pi \left[\frac{y^2}{2} + \frac{4}{3}y^{3/2} - \frac{y^5}{5} - \frac{2y^3}{3} \right] \Big|_0^1 = \pi \left[\frac{1}{2} + \frac{4}{3} - \frac{1}{5} - \frac{2}{3} \right] = \pi \left[\frac{15}{30} + \frac{40}{30} - \frac{6}{30} - \frac{20}{30} \right] = \frac{29\pi}{30} \end{aligned}$$

$$\begin{aligned} \text{By cylindrical shells: } & \int_{x=0}^{x=1} 2\pi(x+1)(\sqrt{x} - x^2) dx = 2\pi \int_0^1 [x^{3/2} + x^{1/2} - x^3 - x^2] dx \\ & = 2\pi \left[\frac{2x^{5/2}}{5} + \frac{2x^{3/2}}{3} - \frac{x^4}{4} - \frac{x^3}{3} \right] \Big|_0^1 = 2\pi \left[\frac{2}{5} + \frac{2}{3} - \frac{1}{4} - \frac{1}{3} \right] = 2\pi \left[\frac{24 + 40 - 15 - 20}{60} \right] = \frac{29\pi}{30} \end{aligned}$$

6. (5 pts) Suppose that a city had a population of 25,000 in 1990 and 30,000 in 2000. If we assume that its population will continue to grow exponentially at a constant rate, what population can its city planners expect in the year 2030? In what year would this model predict a population of 50,000?

We measure time beginning at 1990. Then the general equation for exponential growth gives us $P(t) = 25000e^{kt}$. When $t = 10$, we have $p(t) = 30000$, so after substituting we get $\frac{6}{5} = e^{10k}$, and therefore $k = (.1) \ln(1.2)$.

$$P(40) = P_0 e^{(.1) \ln(1.2) \cdot 40} = P_0 (e^{\ln(1.2)})^4 = 25000(1.2)^4 = 25000 \cdot 2.0736 = 51840$$

$$\text{Solving } P(t) = 50000 \text{ leads to } 2 = e^{kt}, \text{ so } t = \frac{\ln 2}{k} = \frac{10 \ln 2}{\ln(1.2)} \sim 38.02 \text{ years}$$

7. (5 pts) Sugar dissolves in water at a rate proportional to the amount still undissolved. If 50 grams of sugar were present initially, and at the end of 5 minutes this amount is reduced to 20 grams, how long will it take until 90% of the sugar is dissolved?

The first sentence of the problem shows that we have exponential decay, and so the solution will have the form $P(t) = P_0 e^{kt}$, with $P_0 = 50$. To find k , we set $t = 5$, so $20 = 50(e^{5k})$, and $k = \frac{1}{5} \ln(.4)$. When 90% of the sugar has dissolved, the amount remaining is $P(t) = .1P_0 = 5$, so we need to solve the equation $P(t) = 5$, or $5 = 50e^{kt}$. We get $\ln(.1) = \frac{1}{5} \ln(.4) t$, so $t = \frac{5 \ln(.1)}{\ln(.4)} \sim 12.56$ hours.