1. (8 pts each) Find the derivative of each function.

(a) $y = \ln(\sin x)$ $\qquad y' = \dfrac{\cos x}{\sin x} = \cot x$

(b) $y = \ln\left(\sqrt[4]{\dfrac{x^2-1}{x^2+1}}\right) = \dfrac{1}{4}\ln(x^2-1) - \dfrac{1}{4}\ln(x^2+1)$ $\qquad y' = \dfrac{1}{4}\cdot\dfrac{2x}{x^2-1} - \dfrac{1}{4}\cdot\dfrac{2x}{x^2+1} = \dfrac{x}{x^4-1}$

(c) $y = \tan\left(e^{\sqrt{x}}\right)$ $\qquad y' = \sec^2\left(e^{\sqrt{x}}\right)\cdot e^{\sqrt{x}}\cdot\dfrac{1}{2}x^{-1/2} = \dfrac{e^{\sqrt{x}}}{2\sqrt{x}}\sec^2\left(e^{\sqrt{x}}\right)$

2. (10 pts each) Find each of the following integrals.

(a) $\displaystyle\int_3^5 \frac{2x}{x^2-5}\,dx = \int_4^{20}\frac{du}{u} = \ln|u|\Big|_4^{20} = \ln 20 - \ln 4 = \ln 5 = 1.6094$, for $u = x^2-5$.

(b) $\displaystyle\int_0^2 xe^{4-x^2}\,dx = -\frac{1}{2}\int_4^0 e^u\,du = \frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}e^u\Big|_0^4 = \frac{1}{2}(e^4-1)$, for $u = 4-x^2$.

(c) $\displaystyle\int\frac{dx}{x\ln x} = \int\frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$, for $u = \ln x$

3. (6 pts) Given $f(x) = x^3-3$, find the derivative of $f^{-1}(x)$ at 5. Setting $f(x) = 5$ gives $x^3 = 8$, so $f^{-1}(5) = 2$. We have $f'(x) = 3x^2$, and since $\dfrac{d}{dx}f^{-1}(x) = \dfrac{1}{f'(f^{-1}(x))}$, the derivative of $f^{-1}(x)$ at 5 is $\dfrac{1}{3(2)^2} = \dfrac{1}{12}$.

4. (10 pts) Find the volume of a hemisphere by revolving the following region about the x-axis: the region is in the first quadrant, bounded by the lines $x = 0$, $y = 0$ and the circle $x^2+y^2 = R^2$.

$$V = \int_{x=0}^{x=R}\pi y^2\,dx = \pi\int_0^R (R^2-x^2)\,dx = \pi\left(R^2x - \frac{x^3}{3}\right)\Bigg|_0^R = \pi\left(R^3 - \frac{1}{3}R^3\right) = \frac{2}{3}\pi R^3$$

5. (20 pts) The region bounded by the curves $y = x^2$ and $x = y^2$ is revolved about the line $x = -1$. Find the volume of the resulting region, using **both** the "washer" method (by slicing) and the method of "cylindrical shells".

By washers: $\displaystyle\pi\int_{x=0}^{x=1}\left[(\sqrt{y}+1)^2 - (y^2+1)^2\right]dy = \pi\int_0^1\left[y + 2y^{1/2} + 1 - y^4 - 2y^2 - 1\right]dy$

$$= \pi\left[\frac{y^2}{2} + \frac{4}{3}y^{3/2} - \frac{y^5}{5} - \frac{2y^3}{3}\right]\Bigg|_0^1 = \pi\left[\frac{1}{2}+\frac{4}{3}-\frac{1}{5}-\frac{2}{3}\right] = \pi\left[\frac{15}{30}+\frac{40}{30}-\frac{6}{30}-\frac{20}{30}\right] = \frac{29\pi}{30}$$

By cylindrical shells: $\displaystyle\int_{x=0}^{x=1}2\pi(x+1)(\sqrt{x}-x^2)\,dx = 2\pi\int_0^1\left[x^{3/2}+x^{1/2}-x^3-x^2\right]dx$

$$= 2\pi\left[\frac{2x^{5/2}}{5}+\frac{2x^{3/2}}{3}-\frac{x^4}{4}-\frac{x^3}{3}\right]\Bigg|_0^1 = 2\pi\left[\frac{2}{5}+\frac{2}{3}-\frac{1}{4}-\frac{1}{3}\right] = 2\pi\left[\frac{24+40-15-20}{60}\right] = \frac{29\pi}{30}$$

6. (5 pts) Suppose that a city had a population of 25,000 in 1990 and 30,000 in 2000. If we assume that its population will continue to grow exponentially at a constant rate, what population can its city planners expect in the year 2030? In what year would this model predict a population of 50,000?

We measure time begining at 1990. Then the general equation for exponential growth gives us $P(t) = 25000e^{kt}$. When $t = 10$, we have $p(t) = 30000$, so after substituting we get $\dfrac{6}{5} = e^{10k}$, and therefore $k = (.1)\ln(1.2)$.

$P(40) = P_0e^{(.1)\ln(1.2)\cdot 40} = P_0(e^{\ln(1.2)})^4 = 25000(1.2)^4 = 25000\cdot 2.0736 = 51840$

Solving $P(t) = 50000$ leads to $2 = e^{kt}$, so $t = \dfrac{\ln 2}{k} = \dfrac{10\ln 2}{\ln(1.2)} \sim 38.02$ years

7. (5 pts) Sugar dissolves in water at a rate proportional to the amount still undissolved. If 50 grams of sugar were present initially, and at the end of 5 minutes this amount is reduced to 20 grams, how long will it take until 90% of the sugar is dissolved?

The first sentence of the problem shows that we have exponential decay, and so the solution will have the form $P(t) = P_0e^{kt}$, with $P_0 = 50$. To find k, we set $t = 5$, so $20 = 50(e^{5k})$, and $k = \dfrac{1}{5}\ln(.4)$. When 90% of the sugar has dissolved, the amount remaining is $P(t) = .1P_0 = 5$, so we need to solve the equation $P(t) = 5$, or $5 = 50e^{kt}$. We get $\ln(.1) = \dfrac{1}{5}\ln(.4)\,t$, so $t = \dfrac{5\ln(.1)}{\ln(.4)} \sim 12.56$ hours.