MATH 230 §4 TEST II NAME SOLUTIONS

Prof. J. Beachy 3/23/2007 No calculators. Show all necessary work.

1. (12 pts) Find the derivative y' of each function.

(a) $y = (x^4+1)\tan^{-1} x$ $\quad y' = (4x^3)(\tan^{-1} x) + (x^4+1)\left(\dfrac{1}{x^2+1}\right)$

(b) $y = \sec^{-1}(4x^3)$ $\quad y' = \left(\dfrac{1}{4x^3\sqrt{(4x^3)^2-1}}\right)(12x^2)$ *You can use the integration formula in reverse.*

2. (18 pts) (a) $\displaystyle\int \sec(4\theta)\tan(4\theta)\,d\theta = \frac{1}{4}\sec(4\theta) + C$

(b) Determine the *form* of the partial fractions decomposition of $\dfrac{x^3-6}{(x-5)^2(2x^2+1)(x+7)}$.

$$\frac{A}{(x-5)^2} + \frac{B}{x-5} + \frac{Cx+D}{2x^2+1} + \frac{E}{x+7}$$

(c) Find a triangle you could use to make a trig substitution to evaluate $\dfrac{4}{\sqrt{x^2+2x+65}}\,dx$, **or** find the substitution directly. *Hint: first complete the square.*

$\sqrt{x^2+2x+65} = \sqrt{x^2+2x+1+64} = \sqrt{(x+1)^2+8^2}$

You need $x+1$ as one side of the triangle and 8 as the other, with $\sqrt{(x+1)^2+8^2}$ as the hypotenuse.

This gives $\tan\theta = \dfrac{x+1}{8}$, which leads to the substitution $x = -1 + 8\tan\theta$.

3. (15 pts) Find each of these limits. Explain your answer.

(a) $\displaystyle\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ Since this is the inverse of the tangent, whose graph has a vertical asymptote at $x = \dfrac{\pi}{2}$, there is a horizontal asymptote at $y = \dfrac{\pi}{2}$.

(b) $\displaystyle\lim_{x\to 0} \frac{e^{3x}-1}{\sin x} = \lim_{x\to 0}\frac{3e^{3x}}{\cos x} = \frac{3}{1} = 3$

L'Hospital's rule can be applied because the limit has the indeterminate form $\dfrac{0}{0}$.

(c) $\displaystyle\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = e^4$

Let $y = (1+\sin 4x)^{\cot x}$, so that $\ln y = \ln\left((1+\sin 4x)^{\cot x}\right) = \cot x \ln(1+\sin 4x) = \dfrac{\ln(1+\sin 4x)}{\tan x}$.

Now when we take the limit we get the indeterminate form $\dfrac{0}{0}$, and so we can apply L'Hospital's rule.

$$\lim_{x\to 0^+}\frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+}\frac{\dfrac{4\cos x}{1+\sin 4x}}{\sec^2 x} = \frac{4}{1} = 4.$$ Since $\displaystyle\lim_{x\to 0^+}\ln y = 4$, we have $\displaystyle\lim_{x\to 0^+} y = e^4$,

4. (5 pts) To find $\displaystyle\lim_{x\to 1}\frac{x^3-x^2}{x^3-1}$ a Math 229 student writes $\displaystyle\lim_{x\to 1}\frac{x^3-x^2}{x^3-1} = \lim_{x\to 1}\frac{x^2}{x^2+x+1} = \frac{1}{3}$

while a Math 230 student writes $\displaystyle\lim_{x\to 1}\frac{x^3-x^2}{x^3-1} = \lim_{x\to 1}\frac{3x^2-2x}{3x^2} = \lim_{x\to 1}\frac{6x-2}{6x} = \frac{2}{3}$.

Which answer is correct? Or are they both wrong? Explain. The correct answer is $\dfrac{1}{3}$.

The first answer comes from factoring and canceling $(x-1)$, since $x^3-x^2 = (x-1)x^2$ and $x^3-1 = (x-1)(x^2+x+1)$.

The second answer comes from an *incorrect* application of L'Hospital's rule. The limit $\displaystyle\lim_{x\to 1}\frac{3x^2-2x}{3x^2}$ is not an indeterminate form: substituting $x = 1$ gives the correct answer at this stage. Since it is not an indeterminate form, you cannot differentiate again.

5. (10 pts) (a) Use integration by parts to find $\int \tan^{-1} x\, dx$.

Let $u = \tan^{-1} x$ and $dv = dx$, so that $du = \dfrac{dx}{x^2+1}$ and $v = x$.

$$\int \tan^{-1} x\, dx = x\tan^{-1} x - \int \frac{x}{x^2+1}\, dx = x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{x^2+1}\, dx = x\tan^{-1} x - \frac{1}{2}\ln(x^2+1) + C.$$

6. (10 pts) $\int \cos^3\theta \sin^2\theta\, d\theta = \int (1-\sin^2\theta)\sin^2\theta\, \cos\theta\, d\theta$

Let $u = \sin\theta$, so that $du = \cos\theta\, d\theta$.

$$\int \cos^3\theta\sin^2\theta\, d\theta = \int (1-u^2)u^2 du = \int (u^2-u^4)du = \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5} + C$$

7. (10 pts) $\int x^4 \ln x\, dx =$ Use integration by parts.

Let $u = \ln x$ and $dv = x^4\, dx$, so that $du = \dfrac{dx}{x}$ and $v = \dfrac{x^5}{5}$.

$$\int x^4 \ln x\, dx = \frac{x^5\ln x}{5} - \int \frac{x^5}{5}\cdot\frac{1}{x}\, dx = \frac{x^5\ln x}{5} - \frac{1}{5}\int x^4\, dx = \frac{x^5\ln x}{5} - \frac{x^5}{25} + C$$

8. (10 pts) $\int \dfrac{2x+1}{x^2-7x+12}\, dx =$ Use partial fractions.

We have $\dfrac{2x+1}{(x-3)(x-4)} = \dfrac{A}{x-3} + \dfrac{B}{x-4}$, so $2x+1 = A(x-4) + B(x-3)$.

If we let $x = 3$ we get $A = -7$, and if we let $x = 4$, we get $B = 9$.

$$\int \frac{2x+1}{x^2-7x+12}\, dx = -7\int \frac{dx}{x-3} + 9\int \frac{dx}{x-4} = -7\ln(x-3) + 9\ln(x-4) + C$$

9. (10 pts) $\int \dfrac{4x^2}{(1-x^2)^{3/2}}\, dx = \int \dfrac{4x^2}{(\sqrt{1-x^2})^3}\, dx$

Use a trig substitution. The triangle to use has 1 as the hypotenuse, and x and $\sqrt{1-x^2}$ as the sides. Then the substitution is $x = \sin\theta$, so $dx = \cos\theta\, d\theta$, and $\sqrt{1-x^2} = \cos\theta$.

$$\int \frac{4x^2}{(\sqrt{1-x^2})^3}\, dx = \int \frac{4\sin^2\theta}{\cos^3\theta}\cos\theta\, d\theta = 4\int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta = 4\int \tan^2\theta\, d\theta = 4\int (\sec^2\theta - 1)\, d\theta$$

$$= 4\tan\theta - 4\theta + C = \frac{4x}{\sqrt{1-x^2}} - 4\sin^{-1} x + C$$