1. (12 pts) Find the derivative $y'$ of each function.

(a) $y = (x^4 + 1) \tan^{-1} x$ 
$$y' = (4x^3)(\tan^{-1} x) + (x^4 + 1) \left( \frac{1}{x^2 + 1} \right)$$

(b) $y = \sec^{-1}(4x^3)$ 
$$y' = -\frac{1}{4x^3 \sqrt{(4x^3)^2 - 1}} \left( 12x^2 \right) \quad \text{You can use the integration formula in reverse.}$$

2. (18 pts) (a) \[ \int \sec(4\theta) \tan(4\theta) \, d\theta = \frac{1}{4} \sec(4\theta) + C \]

(b) Determine the form of the partial fractions decomposition of \[ \frac{x^3 - 6}{(x - 5)^2(2x^2 + 1)(x + 7)} \]
\[ \frac{A}{(x - 5)^2} + \frac{B}{x - 5} + \frac{Cx + D}{2x^2 + 1} + \frac{E}{x + 7} \]

(c) Find a triangle you could use to make a trig substitution to evaluate \[ \int \frac{4}{\sqrt{x^2 + 2x + 65}} \, dx, \]
or find the substitution directly. Hint: first complete the square.
\[ \sqrt{x^2 + 2x + 65} = \sqrt{x^2 + 2x + 1 + 64} = \sqrt{(x + 1)^2 + 8^2} \]
You need $x + 1$ as one side of the triangle and 8 as the other, with $\sqrt{(x + 1)^2 + 8^2}$ as the hypotenuse.

This gives $\tan \theta = \frac{x + 1}{8}$, which leads to the substitution $x = -1 + 8 \tan \theta$.

3. (15 pts) Find each of these limits. Explain your answer.

(a) $\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$ 
Since this is the inverse of the tangent, whose graph has a vertical asymptote at $x = \frac{\pi}{2}$, there is a horizontal asymptote at $y = \frac{\pi}{2}$.

(b) $\lim_{x \to 0} \frac{e^{3x} - 1}{\sin x} = \lim_{x \to 0} \frac{3e^{3x}}{\cos x} = \frac{3}{1} = 3$

L’Hospital’s rule can be applied because the limit has the indeterminate form $\frac{0}{0}$.

(c) $\lim_{x \to 0^+} (1 + \sin 4x) \cot x = e^4$
Let $y = (1 + \sin 4x) \cot x$, so that $\ln y = \ln ((1 + \sin 4x) \cot x) = \cot x \ln(1 + \sin 4x) = \frac{\ln(1 + \sin 4x)}{\tan x}$.
Now when we take the limit we get the indeterminate form $\frac{0}{0}$, and so we can apply L’Hospital’s rule.
\[ \lim_{x \to 0^+} \frac{e^{3x} - x^2}{x^3 - 1} \quad \text{which leads to the substitution} \quad x = 1 \]

4. (5 pts) To find $\lim_{x \to 1} \frac{x^3 - x^2}{x^3 - 1}$ a Math 229 student writes
\[ \lim_{x \to 1} \frac{x^3 - x^2}{x^3 - 1} = \lim_{x \to 1} \frac{x^2 - 2x}{3x^2 - 3} = \lim_{x \to 1} \frac{6x - 2}{6x} = 2 \]

while a Math 230 student writes
\[ \lim_{x \to 1} \frac{x^3 - x^2}{x^3 - 1} = \lim_{x \to 1} \frac{3x^2 - 2x}{3x^2} = \lim_{x \to 1} \frac{6x - 2}{6x} = \frac{2}{3} \]

Which answer is correct? Or are they both wrong? Explain. The correct answer is $\frac{1}{3}$.

The first answer comes from factoring and canceling $(x - 1)$, since $x^3 - x^2 = (x - 1)x^2$ and $x^3 - 1 = (x - 1)(x^2 + x + 1)$.

The second answer comes from an incorrect application of L’Hospital’s rule. The limit $\lim_{x \to 1} \frac{3x^2 - 2x}{3x^2}$ is not an indeterminate form: substituting $x = 1$ gives the correct answer at this stage. Since it is not an indeterminate form, you cannot differentiate again.
5. (10 pts) (a) Use integration by parts to find \( \int \tan^{-1} x \, dx \).

Let \( u = \tan^{-1} x \) and \( dv = dx \), so that \( du = \frac{dx}{x^2 + 1} \) and \( v = x \).

\[
\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{x^2 + 1} \, dx = x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{x^2 + 1} \, dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + C.
\]

6. (10 pts) \( \int \cos^3 \theta \sin^2 \theta \, d\theta = \int (1 - \sin^2 \theta) \sin^2 \theta \cos \theta \, d\theta \)

Let \( u = \sin \theta \), so that \( du = \cos \theta \, d\theta \).

\[
\int \cos^3 \theta \sin^2 \theta \, d\theta = \int (1 - u^2) u^2 \, du = \int (u^3 - u^5) \, du = \frac{u^4}{4} - \frac{u^6}{6} + C = \frac{\sin^4 \theta}{4} - \frac{\sin^6 \theta}{6} + C
\]

7. (10 pts) \( \int x^4 \ln x \, dx = \) Use integration by parts.

Let \( u = \ln x \) and \( dv = x^4 \, dx \), so that \( du = \frac{dx}{x} \) and \( v = \frac{x^5}{5} \).

\[
\int x^4 \ln x \, dx = \frac{x^5 \ln x}{5} - \int \frac{x^5 \ln x}{5} \cdot \frac{1}{x} \, dx = \frac{x^5 \ln x}{5} - \frac{1}{5} \int x^4 \, dx = \frac{x^5 \ln x}{5} - \frac{x^5}{25} + C
\]

8. (10 pts) \( \int \frac{2x + 1}{x^2 - 7x + 12} \, dx = \) Use partial fractions.

We have \( \frac{2x + 1}{(x - 3)(x - 4)} = \frac{A}{x - 3} + \frac{B}{x - 4} \), so \( 2x + 1 = A(x - 4) + B(x - 3) \).

If we let \( x = 3 \) we get \( A = -7 \), and if we let \( x = 4 \), we get \( B = 9 \).

\[
\int \frac{2x + 1}{x^2 - 7x + 12} \, dx = -7 \int \frac{dx}{x - 3} + 9 \int \frac{dx}{x - 4} = -7 \ln(x - 3) + 9 \ln(x - 4) + C
\]

9. (10 pts) \( \int \frac{4x^2}{(1 - x^2)^{3/2}} \, dx = \int \frac{4x^2}{(\sqrt{1 - x^2})^3} \, dx \)

Use a trig substitution. The triangle to use has 1 as the hypotenuse, and \( x \) and \( \sqrt{1 - x^2} \) as the sides. Then the substitution is \( x = \sin \theta \), so \( dx \) = \( \cos \theta \, d\theta \), and \( \sqrt{1 - x^2} = \cos \theta \).

\[
\int \frac{4x^2}{(\sqrt{1 - x^2})^3} \, dx = \int \frac{4\sin^2 \theta}{\cos^3 \theta} \cos \theta \, d\theta = 4 \int \frac{\sin^2 \theta}{\cos^2 \theta} \, d\theta = 4 \int \tan^2 \theta \, d\theta = 4 \int (\sec^2 \theta - 1) \, d\theta = 4 \tan \theta - 4\theta + C = \frac{4x}{\sqrt{1 - x^2}} - 4 \ln(x + 1) + C
\]