MATH 240 **TEST I Repeat** 10/2/2002

Prof. John Beachy Show all of the work necessary to justify your answers.

1. (10 pts) If A is a symmetric matrix, show that the matrix $A^2 - 3A + I$ is also symmetric, where I is the identity matrix.

   First, if A is symmetric, then $(A^2)^T = (AA)^T = A^T A^T = AA = A^2$, and so A^2 is symmetric. (Note: in general, if A and B are symmetric, it may not be true that AB is symmetric.)

   Thus if A is symmetric we have the following calculation:
   $(A^2 - 3A + I)^T = (A^2)^T - 3A^T + I^T = A^2 - 3A + I$.

2. (30 pts) Determine whether the given subset W is a subspace of the given vector space V. (In each part, either check that all three of the necessary conditions hold, or give a numerical counterexample to one of them.)

   (a) Let $V = \mathbf{R}^3$ and let $W = \{(x, y, z) \mid z = x^2 + y^2\}$.

   W is the graph of a paraboloid, not a plane, so you should not expect it to be a subspace. It is not closed under scalar multiplication, for example, $(1, 0, 1) \in W$ but $2 \cdot (1, 0, 1) = (2, 0, 2) \notin W$.

   (b) Let $V = \mathcal{C}(-\infty, \infty)$ be the vector space of all real-valued continuous functions defined on the set $\mathbf{R}$ of real numbers. Let W be the set of all functions f in V for which $f(1) = 0$.

   W is a subspace.

   The zero function z belongs since $z(x) = 0$ for all x.

   If $f, g \in W$, then $f(1) = 0$ and $g(1) = 0$, so for the sum of the functions we have $[f + g](1) = f(1) + g(1) = 0 + 0 = 0$, and therefore $f + g \in W$.

   If $f \in W$ and $r \in \mathbf{R}$, then $[rf](1) = r(f(1)) = r \cdot 0 = 0$, and so $rf \in W$.

   (c) Let $V = \mathbf{R}^n$, let A be an $m \times n$ matrix, and let W be all vectors $\mathbf{x}$ in V with $A\mathbf{x} = \mathbf{0}$.

   For the solution, see Example 9 on page 108.

3. (20 pts) In $\mathbf{R}^2$, use ordinary addition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$, but define a new scalar multiplication by $r \cdot (x, y) = (r^2 x, r^2 y)$. Check the distributive law $(r + s) \cdot \mathbf{v} = r \cdot \mathbf{v} + s \cdot \mathbf{v}$ and the associative law $r \cdot (s \cdot \mathbf{v}) = (rs) \cdot \mathbf{v}$. If the law is valid, give a proof. If not, give a numerical counterexample to show that it fails.

   $(r + s) \cdot (x, y) = ((r + s)^2 x, (r + s)^2 y)$, while $r \cdot (x, y) + s \cdot (x, y) = ((r^2 + s^2)x, (r^2 + s^2)^y)$, so choosing $r = 1$, $s = 1$, $x = 1$, $y = 0$ gives a counterexample.

   $r \cdot (s \cdot (x, y)) = r \cdot (s^2 x, s^2 y) = (r^2 s^2 x, r^2 s^2 y)$, while $(rs) \cdot (x, y) = ((rs)^2 x, (rs)^2 y)$, so these are equal for all r, s, x, y and therefore this law holds.

4. (20 pts) For the vectors $\mathbf{v}_1 = t^3 - t^2 + t + 2$, $\mathbf{v}_2 = 2t^3 - 2t^2 + 2t + 4$, $\mathbf{v}_3 = t^2 + t + 2$, and $\mathbf{v}_4 = 2t^3 - t^2 + 4t + 7$ in the vector space P_3, solve the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 = \mathbf{0} \ .$$

Substituting for the given vectors, we have the following equations.

$$x_1(t^3 - t^2 + t + 2) + x_2(2t^3 - 2t^2 + 2t + 4) + x_3(t^2 + t + 2) + x_4(2t^3 - t^2 + 4t + 7) = 0$$

$$(x_1 + 2x_2 + 2x_4)t^3 + (-x_1 - 2x_2 + x_3 - x_4)t^2 + (x_1 + 2x_2 + x_3 + 4x_4)t + (2x_1 + 4x_2 + 2x_3 + 7x_4) = 0$$

A polynomial is equal to the zero polynomial if and only if all of its coefficients are zero, so we get the following system of equations.

$$\begin{array}{rrrrcl} x_1 & +2x_2 & & +2x_4 & = & 0 \\ -x_1 & -2x_2 & +x_3 & -x_4 & = & 0 \\ x_1 & +2x_2 & +x_3 & +4x_4 & = & 0 \\ 2x_1 & +4x_2 & +2x_3 & +7x_4 & = & 0 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ -1 & -2 & 1 & -1 & 0 \\ 1 & 2 & 1 & 4 & 0 \\ 2 & 4 & 2 & 7 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row reducing gives the solution $x_1 = -2x_2$, $x_3 = 0$, $x_4 = 0$.

5. (20 pts) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$. Find A^{-1} and write A as a product of elementary matrices.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \underset{\substack{R_2 - R_1 \\ R_3 - R_1}}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right] \underset{\frac{1}{2}R_2}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\underset{R_3 - 3R_2}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -3/2 & 1 \end{array}\right] \underset{R_1 - R_3}{\rightsquigarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 3/2 & -1 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1/2 & -3/2 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1/2 & 3/2 & -1 \\ -1/2 & 1/2 & 0 \\ 1/2 & -3/2 & 1 \end{bmatrix}$$

We used the following row operations:

$R_2 - R_1$, $R_3 - R_1$, $\frac{1}{2}R_2$, $R_3 - 3R_2$, $R_1 - R_3$.

To express A as a product of elementary matrices, we need to use their inverses:

$R_2 + R_1$, $R_3 + R_1$, $2R_2$, $R_3 + 3R_2$, $R_1 + R_3$.

$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$