1. (20 pts) (a) When solving a system of linear equations, explain why adding a multiple of one equation to another does not change the solution set.

In words: A solution to the original system is still a solution to the new system, since we have just added equal numbers to each side of one of the equations. It is not so obvious that a solution to the new system is still a solution to the original system, but this can be explained by saying that the operation can be reversed, and then the previous argument can be used.

Using symbols: Suppose that X_0 is a solution to the system $AX = B$. Any elementary row operation is represented by an elementary matrix E, and the new system is $(EA)X = EB$, so we can just multiply the equation $AX_0 = B$ on the left by E to show that X_0 is still a solution of the new system. On the other hand, if X_1 is a solution of the new system, we can multiply both sides of the equation $(EA)X_1 = EB$ by E^{-1} to show that $AX_1 = B$, and so X_1 is a solution of the original system.

(b) Use Gauss-Jordan reduction to solve the following linear system.

$$\begin{array}{rrrrcr} x_1 & +2x_2 & & +2x_4 & = & 1 \\ -x_1 & -2x_2 & +x_3 & -x_4 & = & -4 \\ x_1 & +2x_2 & +x_3 & +4x_4 & = & -3 \\ 2x_1 & +4x_2 & +2x_3 & +7x_4 & = & -5 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & -1 & -4 \\ 1 & 2 & 1 & 4 & -3 \\ 2 & 4 & 2 & 7 & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 & -4 \\ 0 & 0 & 2 & 3 & -7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The final solution is that x_2 is an independent variable, with $x_1 = 3 - 2x_2$, $x_3 = -2$, and $x_4 = -1$.

Comments: I feel that you need to practice enough row reductions to become fast and accurate. Don't write down more information than I have written above–it takes too much time to copy the same terms over and over again.

2. (10 pts) Find the inverse of the matrix $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, where a, b, c are any real numbers.

$$\left[\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & b-ac & 1 & -a & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & -b+ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

The final answer is $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -a & -b+ac \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$.

3. (10 pts) Express the matrix $\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$ as a product of elementary matrices.

The first step is to row-reduce the matrix, while keeping track of the elementary matrices that correspond to the operations we have used.

$\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ The corresponding elementary matrices are $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, and $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$. We need to take the inverse of each of these matrices, because the row reduction $E_3E_2E_1A = I$ gives us $A = E_1^{-1}E_2^{-1}E_3^{-1}$.

The final answer is $\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$.

4. (10 pts) If A is a nonsingular $n \times n$ matrix, show that A^T is also nonsingular, and find a formula for the inverse of A^T in terms of the inverse of A.

If A is nonsingular, then it has an inverse A^{-1} with $AA^{-1} = I$, where I is the identity matrix. We can take the transpose of both sides of this equation, giving $(AA^{-1})^T = I^T$, and then simplifying gives $(A^{-1})^T A^T = I$, which shows that A^T is nonsingular. This equation also tells us that the inverse of A^T is $(A^{-1})^T$, and so we have the formula $(A^T)^{-1} = (A^{-1})^T$.

5. (10 pts) Let A, B be nonsingular $n \times n$ matrices.

(a) Given $AB = BA$, prove that $(AB)^{-1} = A^{-1}B^{-1}$.

We can take the inverse of both sides of the equation and simplify, to get $(AB)^{-1} = (BA)^{-1} = A^{-1}B^{-1}$.

(b) Given $(AB)^{-1} = A^{-1}B^{-1}$, prove that $AB = BA$.

Again, we can take the inverse of both sides of the given equation. Thus $((AB)^{-1})^{-1} = (A^{-1}B^{-1})^{-1}$, and so $AB = (B^{-1})^{-1}(A^{-1})^{-1} = BA$.

6. (30 pts) Determine whether the given subset W is a subspace of the vector space V. (In each part, either check that all three of the necessary conditions hold, or give a numerical counterexample to one of them.)

(a) Let $V = \mathbf{R}^3$ and let $W = \{(x, y, z) \mid y = 2x - z\}$.

The set W is a subspace. The zero vector belongs; if (x_1, y_1, z_1) and (x_2, y_2, z_2) belong, then we have $y_1 = 2x_1 - z_1$ and $y_2 = 2x_2 - z_2$. Their sum is $(x_1 + x_2, y_1 + y_2, z_1 + z_2)$, and $y_1 + y_2 = (2x_1 - z_1) + (2x_2 - z_2) = 2(x_1 + x_2) - (z_1 + z_2)$, which shows that the sum belongs to W. For any scalar r, the product is $r(x_1, y_1, z_1) = (rx_1, ry_1, rz_1)$, and $ry_1 = r(2x_1 - z_1) = 2(rx_1) - rz_1$, which shows that the scalar product also belongs to W.

(b) Let $V = \mathbf{R}^2$ and let $W = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

This is not a subspace. The vector $(1, 0)$ belongs to W, but $2(1, 0)$ does not.

(c) Let V be the vector space P_3 of all polynomials of degree at most 3. Let W be the set of all polynomials $p(x)$ in P_3 for which $\int_0^1 p(x)\,dx = 0$.

The zero polynomial belongs, since its integral is certainly 0. Next, suppose that $p(x)$ and $q(x)$ belong to W. Then $\int_0^1 (p(x) + q(x))\,dx = \int_0^1 p(x)\,dx + \int_0^1 q(x)\,dx = 0 + 0 = 0$, and for any scalar k we have $\int_0^1 kp(x)\,dx = k\int_0^1 p(x)\,dx = k \cdot 0 = 0$, so $p(x) + q(x)$ and $kp(x)$ belong to W, showing that W is a subspace.

7. (10 pts) In $\mathbf{R}^2$, use ordinary addition $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\, y_1 + y_2)$, but define a new scalar multiplication by $r \cdot (x, y) = (rx, -ry)$. Check each of the four laws M_1, M_2, M_3, M_4 for scalar multiplication. If the law is valid, give a proof. If not, give a numerical counterexample to show that it is fails.

Axiom M_1 holds since $r \cdot ((x_1, y_1) + (x_2, y_2)) = r \cdot (x_1 + x_2,\, y_1 + y_2) = (r(x_1 + x_2),\, -r(y_1 + y_2))$ and $r \cdot (x_1, y_1) + r \cdot (x_2, y_2) = (rx_1, -ry_1) + (rx_2, -ry_2) = (rx_1 + rx_2, -ry_1 - ry_2)$.

Axiom M_2 holds since $(r + s) \cdot (x_1, y_1) = ((r + s)x_1, -(r + s)y_1)$ and $r \cdot (x_1, y_1) + s \cdot (x_1, y_1) = (rx_1, -ry_1) + (sx_1, -sy_1) = (rx_1 + sx_1, -ry_1 - sy_1)$.

Axiom M_3 does not hold since $(1 \cdot 2) \cdot (3, 4) = 2 \cdot (3, 4) = (6, -8)$ but $1 \cdot (2 \cdot (3, 4)) = 1 \cdot (6, -8) = (6, 8)$.

Axiom M_4 does not hold since $1 \cdot (0, 1) = (0, -1) \neq (0, 1)$.