1. (20 pts; 4.2 #16) Let $L : \mathbf{R}^5 \to \mathbf{R}^4$ be the linear transformation defined by $L(\mathbf{x}) = A\mathbf{x}$, for the matrix $A = \begin{bmatrix} 1 & 0 & -1 & 3 & -1 \\ 1 & 0 & 0 & 2 & -1 \\ 2 & 0 & -1 & 5 & -1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}$. Given that A row-reduces to $\begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

(a) [10 pts] find a basis for $\ker L$;

Since $\ker L$ is the solution space of the equation $L(\mathbf{x}) = \mathbf{0}$, we need to find the solution space of the system of equations $A\mathbf{x} = \mathbf{0}$. After row-reducing the matrix, we have the corresponding equations $x_1 = -2x_4$; $x_3 = x_4$; $x_5 = 0$. Choose $x_2 = 1$, $x_4 = 0$ and then $x_2 = 0$, $x_4 = 1$ to obtain the basis $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

(b) [6 pts] find a basis for $\operatorname{range} L$;

Since $\operatorname{range} L$ is the column space of A, as basis we take the columns of A that correspond to the leading 1's in the reduced matrix, giving us the basis $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}$.

(c) [4 pts] find $\dim(\ker L)$ and $\dim(\operatorname{range} L)$.

We have $\dim(\ker L) = 2$ and $\dim(\operatorname{range} L) = 3$. Check: $2 + 3 = \dim(\mathbf{R}^5)$.

2. (20 pts; compare 4.5 #2) Let $L : \mathbf{R}_3 \to \mathbf{R}_3$ be the linear transformation defined by $L(x_1, x_2, x_3) = (x_1,\ 2x_2 + x_3,\ x_2 + 2x_3)$. Let $S = \{(1,0,0),\ (0,1,0),\ (0,0,1)\}$ be the standard basis for $\mathbf{R}_3$, and let T be the basis $\{(1,0,0),\ (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}),\ (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$ for $\mathbf{R}_3$.

(a) [5 pts] Find the matrix representation $M_{S\leftarrow S}(L)$ of L with respect to the basis S.

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ 2x_2 + x_3 \\ x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad M_{S\leftarrow S}(L) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

(b) [7 pts] Find the matrix representation $M_{T\leftarrow T}(L)$ of L with respect to the basis T.

$$[\,L(T)\,] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$[\,T \mid L(T)\,] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 3 & -1 \\ 0 & 1 & 1 & 0 & 3 & 1 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 2 & 0 & 0 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \qquad M_{T\leftarrow T}(L) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notes: If you realize that $L(\mathbf{v}_1) = \mathbf{v}_1, L(\mathbf{v}_2) = 3\mathbf{v}_2, L(\mathbf{v}_3) = \mathbf{v}_3$, for the new basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, then you can immediately write down the matrix $M_{T\leftarrow T}(L)$. Actually, the new basis was chosen to do exactly this.

The matrix came from Exercise 15 in Section 6.8. You now have enough background to read Sections 6.7 and 6.8, which give some interesting applications of the theory of similar matrices in geometry. For certain curves in $\mathbf{R}^2$ and surfaces in $\mathbf{R}^3$, finding a similar diagonal matrix allows us to put a curve or surface into standard form by choosing an appropriate basis. Diagonalizing this particular matrix allows us to see that the quadric surface $x^2 + 2y^2 + 2z^2 + 2yz = 1$ is an ellipsoid which is written in standard form as

$$\frac{x'^2}{1} + \frac{y'^2}{1} + \frac{z'^2}{1/3} = 1$$

after making the substitution $x = -y'$, $y = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}z'$, $z = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{2}}z'$.

(c) [5 pts] Find the transition matrices $P_{S\leftarrow T}$ and $P_{T\leftarrow S}$ that change coordinates.

Since S is the standard basis, $P_{S\leftarrow T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, and $P_{T\leftarrow S} = (P_{S\leftarrow T})^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & \sqrt{2} & 0 \\ 0 & 1 & 1 & 0 & 0 & \sqrt{2} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 2 & 0 & -\sqrt{2} & \sqrt{2} \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right] \qquad P_{T\leftarrow S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Notes: Finding the inverse of the matrix $P_{S\leftarrow T}$ can be reduced to finding the inverse of a 2×2 matrix if you remember that for matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ in block form, we have $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}$, provided A^{-1} and B^{-1} exist.

If you recognize that $P_{S\leftarrow T}$ is an orthogonal matrix (since T happens to be an orthonormal basis), then to find $P_{T\leftarrow S}$ you only need to take the transpose, since the transpose of an orthogonal matrix is its inverse.

(d) [3 pts] Check that $M_{T\leftarrow T}(L) = P_{T\leftarrow S}\cdot M_{S\leftarrow S}(L)\cdot P_{S\leftarrow T}$.

$$P_{T\leftarrow S}\cdot M_{S\leftarrow S}(L)\cdot P_{S\leftarrow T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = M_{T\leftarrow T}(L)$$

3. (20 pts; compare 4.3 #11) Let M_{22} be the vector space of all 2×2 matrices, and let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Define $L : M_{22} \to M_{22}$ by $L(X) = AX - XA$, for $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. Let $S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$, $T = \left\{\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$.

(a) [6 pts] Check that L is a linear transformation.

$L(X+Y) = A(X+Y) - (X+Y)A = AX + AY - XA - YA = AX - XA + AY - YA = L(X) + L(Y)$
$L(cX) = A(cX) - (cX)A = c(AX - XA) = cL(X)$

(b) [6 pts] Find the matrix representation $M_{S\leftarrow S}(L)$ of L with respect to the basis S.

$$L\left(\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} - \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+x_3 & x_2+x_4 \\ x_3 & x_4 \end{bmatrix} - \begin{bmatrix} x_1 & x_1+x_2 \\ x_3 & x_3+x_4 \end{bmatrix} = \begin{bmatrix} x_3 & x_4-x_1 \\ 0 & -x_3 \end{bmatrix}$$

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_4-x_1 \\ 0 \\ -x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad M_{S\leftarrow S}(L) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

(c) [8 pts] Find the matrix representation $M_{T\leftarrow T}(L)$ of L with respect to the basis T.

$$[\ L(T)\ ] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$[\,T \mid L(T)\,] = \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \end{array}\right] \rightsquigarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \qquad M_{T\leftarrow T}(L) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Notes: As in problem #2, the new basis was carefully chosen. It puts the matrix $M_{T\leftarrow T}(L)$ into what is called Jordan canonical form, in which the eigenvalues of the matrix appear on the main diagonal, and 0's and 1's appear just above the main diagonal. In this case 0 *is the only eigenvalue of the matrix.*

4. (10 pts; 5.3 # 12) Find all values of x for which $\begin{vmatrix} x-1 & 0 & 1 \\ -2 & x+2 & -1 \\ 0 & 0 & x+1 \end{vmatrix} = 0.$

$$\begin{vmatrix} x-1 & 0 & 1 \\ -2 & x+2 & -1 \\ 0 & 0 & x+1 \end{vmatrix} = (x-1)\begin{vmatrix} x+2 & -1 \\ 0 & x+1 \end{vmatrix} - 0\begin{vmatrix} -2 & -1 \\ 0 & x+1 \end{vmatrix} + 1\begin{vmatrix} -2 & x+2 \\ 0 & 0 \end{vmatrix}$$

$= (x-1)(x+2)(x+1)$ so $(x-1)(x+2)(x+1) = 0$ has the solution $x = 1, -2, -1$.

5. (10 pts; compare 5.4 #10) Find the adjoint of $A = \begin{bmatrix} a & x & 0 \\ 0 & a & y \\ 0 & 0 & a \end{bmatrix}$, and then find A^{-1}, when $a \neq 0$.

Matrix of cofactors: $\begin{bmatrix} a^2 & 0 & 0 \\ -ax & a^2 & 0 \\ xy & -ay & a^2 \end{bmatrix}$ $\qquad$ $\text{adj}(A) = \begin{bmatrix} a^2 & -ax & xy \\ 0 & a^2 & -ay \\ 0 & 0 & a^2 \end{bmatrix}$

[3 pts] $A^{-1} = \dfrac{1}{\det(A)}\text{adj}(A) = \begin{bmatrix} \frac{1}{a} & -\frac{x}{a^2} & \frac{xy}{a^3} \\ 0 & \frac{1}{a} & -\frac{y}{a^2} \\ 0 & 0 & \frac{1}{a} \end{bmatrix}$

6. (10 pts; 5.2 #11) Recall that the matrix A is skew-symmetric if $A^T = -A$. Explain why $\det(A) = 0$ if A is a skew-symmetric $n \times n$ matrix, where n is odd. Give an example of a 2×2 skew-symmetric matrix whose determinant is *not* zero.

$A^T = -A$ so $\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$

If n is odd, then $\det(A) = -\det(A)$, and therefore $\det(A) = 0$.

[3 pts] Example: $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is skew-symmetric with determinant 1.

7. (10 pts; 4.5 #7) Show that if the $n \times n$ matrix B is similar to matrix A, then B^T is similar to A^T.

If B is similar to A, then $B = P^{-1}AP$ for some invertible matrix P. Taking the transpose gives $B^T = (P^{-1}AP)^T = P^T A^T (P^{-1})^T$. Since $P^T(P^{-1})^T = (P^{-1}P)^T = I^T = I$, we can see that P^T is the inverse of $(P^{-1})^T$. Thus $B^T = X^{-1}A^T X$ for $X = (P^{-1})^T$, and this shows that B^T is similar to A^T.

Grades: 90–100 A (5); 74–84 B (8); 52–71 C (10); 47–48 D (2).