MATH 240 **TEST III** **November 14, 1997**

Prof. John Beachy Show all of the work necessary to justify your answers.

1. (15 pts) Define a linear transformation $L : P_3 \to P_5$ by $L(p(t)) = p(t)(t^2 - 3t + 2)$, for all polynomials $p(t)$ in P_3.

(a) Find the matrix of L relative to the ordered bases $S = \{t^3, t^2, t, 1\}$ for P_3 and $T = \{t^5, t^4, t^3, t^2, t, 1\}$ for P_5.

(b) Find the dimension of $range(L)$ and the dimension of $ker(L)$.

2. (25 pts) Remember that M_{nn} is the vector space of $n \times n$ matrices.

(a) Let A be a fixed $n \times n$ matrix, and define $L : M_{nn} \to M_{nn}$ by $L(X) = AX - XA$, for all $n \times n$ matrices X. Check that L defines a linear transformation.

(b) For the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, find the matrix of the transformation $L : M_{22} \to M_{22}$ defined in part (a), relative to the basis $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(c) Find a basis for $range(L)$ and a basis for $ker(L)$ (these should consist of matrices in M_{22}).

3. (20 pts) Define the linear transformation $L : \mathbf{R}^2 \to \mathbf{R}^2$ by $L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$.

(a) Find the matrix of L relative to the standard basis S for $\mathbf{R}^2$.

(b) Find the matrix of L relative to the basis $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ by using transition matrices.

(c) Find the matrix of L relative to the basis T by direct computation, and compare your answer with the one you got in part (b).

4. (20 pts) (a) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be a basis for an inner product space V. Write out the formulas necessary to produce the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in the first step of the Gram–Schmidt process.

(b) In using the Gram–Schmidt process to transform $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ into an orthonormal basis for $\mathbf{R}^3$ (using the inner product defined by the ordinary dot product), the first two steps produce the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. *Use the Gram–Schmidt process to find the third vector* $\mathbf{v_3}$*, and then find the corresponding orthonormal basis* $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

5. (10 pts) Let V be an innner product space, with inner product $(\mathbf{u}, \mathbf{v})$. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be linearly independent vectors in V. Show that if $\mathbf{u}$ is a nonzero vector in V that is orthogonal to each of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, then the set $\{\mathbf{u}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of vectors.

6. (10 pts) Let P be the vector space of all polynomials with real coefficients. Show that all of the conditions for an inner product hold when we define

$$(p(t), q(t)) = \int_0^1 p(t)q(t)dt \; ,$$

for polynomials $p(t), q(t)$ in P.