MATH 240 HOMEWORK 5 NAME ____________________

Prof. J. Beachy Friday, 10/18/02 Score / 20

1. (p. 211 #14) Let V be the vector space of all polynomials, and define on it the inner product $(p(t), q(t)) = \int_0^1 p(t)q(t)dt$. Find the cosine of the angle between the following pairs of vectors in V.

(a) $p(t) = t$, $q(t) = t - 1$.

Since $\cos\theta = \dfrac{(\mathbf{u}, \mathbf{v})}{||\mathbf{u}|| \cdot ||\mathbf{v}||}$, we need to compute

$(p(t), q(t)) = \int_0^1 p(t)q(t)dt = \int_0^1 t(t-1)dt = \int_0^1 (t^2 - t)dt = \frac{t^3}{3} - \frac{t^2}{2}\Big|_0^1 = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$

$(p(t), p(t)) = \int_0^1 p(t)^2 dt = \int_0^1 t^2 dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3} = \frac{1}{3}$

$(q(t), q(t)) = \int_0^1 q(t)^2 dt = \int_0^1 (t^2 - 2t + 1)dt = \frac{t^3}{3} - t^2 + t\Big|_0^1 = \frac{1}{3} - 1 + 1 = \frac{1}{3}$

$\cos\theta = -\frac{1}{2}$

(b) $p(t) = t$, $q(t) = t$.

In this case, $\cos\theta = \dfrac{(p(t), p(t))}{||p(t)|| \cdot ||p(t)||} = \dfrac{(p(t), p(t))}{(p(t), p(t))} = 1$. Better: $\theta = 0$ so $\cos\theta = 1$.

(c) $p(t) = 1$, $q(t) = 2t + 3$.

$(p(t), q(t)) = \int_0^1 p(t)q(t)dt = \int_0^1 (2t+3)dt = t^2 + 3t\Big|_0^1 = 4$

$(p(t), p(t)) = \int_0^1 p(t)^2 dt = \int_0^1 dt = t|_0^1 = 1$

$(q(t), q(t)) = \int_0^1 q(t)^2 dt = \int_0^1 (4t^2 + 12t + 9)dt = \frac{4t^3}{3} + 6t^2 + 9t\Big|_0^1 = \frac{49}{3}$

$\cos\theta = \dfrac{(1, 2t+3)}{||1|| \cdot ||2t+3||} = \dfrac{4}{1 \cdot \sqrt{49/3}} = \dfrac{4\sqrt{3}}{7}$

2. (p 224 #10) Use the Gram–Schmidt process to transform the basis $\{\mathbf{u}_1 = (1,1,1),\ \mathbf{u}_2 = (0,1,1),\ \mathbf{u}_3 = (1,2,3)\}$ for $\mathbf{R}_3$ into an orthonormal basis for $\mathbf{R}_3$.

Let $\mathbf{v}_1 = (1,1,1)$.

Then $\mathbf{v}_2 = \mathbf{u}_2 - \dfrac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (0,1,1) - \dfrac{(0,1,1)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) = (0,1,1) - \dfrac{2}{3}(1,1,1) = (1/3)(-2,1,1)$

Let $\mathbf{v}_2 = (-2,1,1)$.

$$\mathbf{v}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1\right) = (1,2,3) - \frac{(1,2,3)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) - \frac{(1,2,3)\cdot(-2,1,1)}{(-2,1,1)\cdot(-2,1,1)}(-2,1,1)$$

$$= (1,2,3) - \frac{6}{3}(1,1,1) - \frac{3}{6}(-2,1,1) = (1,2,3) - (2,2,2) - (-1,1/2,1/2) = (1/2)(0,-1,1)$$

Let $\mathbf{v}_3 = (0,-1,1)$.

The final step is to normalize by dividing by the lengths.

$\mathbf{w}_1 = \dfrac{1}{\sqrt{3}}(1,1,1)$; $\quad \mathbf{w}_2 = \dfrac{1}{\sqrt{6}}(-2,1,1)$; $\quad \mathbf{w}_3 = \dfrac{1}{\sqrt{2}}(0,-1,1)$.