MATH 240 HOMEWORK 7 SOLUTIONS NAME ____________________

Prof. J. Beachy Wednesday, 11/13/02 Score / 20

1. Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2 \\ x_1-x_2 \\ x_1+x_3 \end{bmatrix}$, for all vectors in $\mathbf{R}^3$.

Let S be the standard basis for $\mathbf{R}^3$, and let T be the basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Find the matrix representation $M_{S\leftarrow S}(L)$ of L with respect to S.

$$M_{S\leftarrow S}(L) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ since } \begin{bmatrix} x_1+x_2 \\ x_1-x_2 \\ x_1+x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) Find the transition matrices $P_{T\leftarrow S}$ and $P_{S\leftarrow T}$.

$P_{S\leftarrow T}$: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ *The algorithm is to form* $[\, S \mid T \,]$ *and row reduce to get* $[\, I \mid P_{S\leftarrow T} \,]$.

$P_{T\leftarrow S}$:
$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \qquad P_{T\leftarrow S} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

(c) Find the matrix representation $M_{T\leftarrow T}(L)$ of L with respect to T directly,
and by computing $P_{T\leftarrow S} \cdot M_{S\leftarrow S}(L) \cdot P_{S\leftarrow T}$.

We've already computed the matrices we need, so $P_{T\leftarrow S} \cdot M_{S\leftarrow S}(L) \cdot P_{S\leftarrow T} =$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

To find $M_{T\leftarrow T}(L)$ directly, the algorithm is to form $[\, T \mid L(T) \,]$ and row reduce to get $[\, I \mid P_{T\leftarrow T} \,]$.

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \quad L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 2 & 2 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 2 & 1 & 1 \\ 0 & -1 & 1 & -2 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$$
$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \qquad M_{T\leftarrow T}(L) = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix}$$

This shows that $P_{T\leftarrow S} \cdot M_{S\leftarrow S}(L) \cdot P_{S\leftarrow T} = M_{T\leftarrow T}(L)$, and the matrix can be calculated either way.

2. Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be defined by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_2 + x_3 \\ x_1 + x_2 + x_3 \end{bmatrix}$, for all vectors in $\mathbf{R}^3$.

Let S be the standard basis for $\mathbf{R}^3$, and let T be the basis $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Find the matrix representation $M_{S\leftarrow S}(L)$ of L with respect to S.

$$M_{S\leftarrow S}(L) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ since } \begin{bmatrix} x_3 \\ x_2 + x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) Find the transition matrices $P_{T\leftarrow S}$ and $P_{S\leftarrow T}$.

$$P_{S\leftarrow T}: \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P_{T\leftarrow S}: \quad \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \qquad P_{T\leftarrow S} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(c) Find the matrix representation $M_{T\leftarrow T}(L)$ of L with respect to T directly,
and by computing $P_{T\leftarrow S} \cdot M_{S\leftarrow S}(L) \cdot P_{S\leftarrow T}$.

$P_{T\leftarrow S} \cdot M_{S\leftarrow S}(L) \cdot P_{S\leftarrow T} =$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

To find $M_{T\leftarrow T}(L)$ directly, the algorithm is to form $[\, T \mid L(T)\,]$ and row reduce to get $[\, I \mid P_{T\leftarrow T}\,]$.

$$L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad L\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 & 2 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \qquad M_{T\leftarrow T}(L) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

This shows that $P_{T\leftarrow S} \cdot M_{S\leftarrow S}(L) \cdot P_{S\leftarrow T} = M_{T\leftarrow T}(L)$. It's interesting that although the two bases were different, we happened to get the same matrices.