

1. (10 points) Determine whether these vectors are linearly dependent or linearly independent (in the vector space of all polynomial functions):

$$\{t^2 + 1, t^2 + t, t - 1\}.$$

We need to solve $x_1(t^2 + 1) + x_2(t^2 + t) + x_3(t - 1) = 0$.

This can be rewritten as $(x_1 + x_2)t^2 + (x_2 + x_3)t + (x_1 - x_3) = 0$.

Since a polynomial is the zero polynomial if and only if all of its coefficients are equal to zero, we get

$$x_1 + x_2 = 0 \quad x_2 + x_3 = 0 \quad x_1 - x_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This shows that there are nontrivial solutions and so the vectors are linearly dependent. (Actually, $x_1 = x_3$, $x_2 = -x_3$, so for $x_3 = -1$ we get $t - 1 = (-1)(t^2 + 1) + (1)(t^2 + t)$.)

2. (10 points) For what values of c are the matrices $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 1 & 1 \\ 0 & c \end{bmatrix}$ linearly independent in the vector space M_{22} ?

We need to solve $x_1 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 1 \\ 0 & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} -x_1 + 2x_2 + x_3 & x_2 + x_3 \\ 0 & -x_1 + 2x_2 + cx_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We get $-x_1 + 2x_2 + x_3 = 0$ $x_2 + x_3 = 0$ $-x_1 + 2x_2 + cx_3 = 0$

$$\left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 2 & c & 0 \end{array} \right] \rightsquigarrow \left[\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & c-1 & 0 \end{array} \right]$$

This shows that the matrices are linearly independent if and only if $c - 1 \neq 0$, or $c \neq 1$.