MATH 240 QUIZ 5 NAME SOLUTIONS

Prof. J. Beachy Friday, 10/4/02 Score / 20

1. (10 pts) You are given a basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ for a vector space V. Define new vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$, and $\mathbf{w}_3 = \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3$. Is the set $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ also a basis for V?

The given basis has 3 elements, so we know that $\dim(V) = 3$. Theorem 2.11 says that to show that $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for V we can either check that the vectors are linearly independent or that they span V (we do *not* have to check both conditions).

To check for linear independence we need to solve the equation $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3 = \mathbf{0}$. We can substitute to get

$x_1(\mathbf{v}_1 + \mathbf{v}_3) + x_2(\mathbf{v}_2 + \mathbf{v}_3) + x_3(\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3) = \mathbf{0}$

$(x_1 + x_3)\mathbf{v}_1 + (x_2 + 2x_3)\mathbf{v}_2 + (x_1 + x_2 - x_3)\mathbf{v}_3 = \mathbf{0}$

Because the vectors $\mathbf{v}_i$ are linearly independent, this implies that $x_1 + x_3 = 0$, $x_2 + 2x_3 = 0$, and $x_1 + x_2 - x_3 = 0$. Solving this system, we have the following matrix reduction.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$$

At this point you can see that the matrix will reduce to the identity, so the only solution is the trivial solution. Conclusion: the vectors $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are linearly independent and therefore form a basis for V.

2. (10 pts) Find a basis for the null space of the matrix $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 2 & 0 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$.

The first step is to row-reduce the matrix.

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 2 & 0 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -4 & 2 \\ 0 & -2 & -8 & 4 \\ 0 & -1 & -4 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -5 & 3 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives us the solution $x_1 = 5x_3 - 3x_4$, $x_2 = -4x_3 + 2x_4$. First set $x_3 = 1, x_4 = 0$, and then set $x_3 = 0, x_4 = 1$. This gives us the following basis for the null space.

$$\begin{bmatrix} 5 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$