MATH 240 QUIZ 6 NAME SOLUTIONS

Prof. J. Beachy Friday, 10/11/02 Score / 20

1. (12 pts) Let $S = \{[1\ \ 2], [0\ \ 1]\}$ and $T = \{[1\ \ 1], [2\ \ 3]\}$.
Find the transition matrix $P_{S\leftarrow T}$ that changes from coordinates relative to T to coordinates relative to S.
For $\mathbf{v} = [1\ \ 3]$, first find $[\mathbf{v}]_S$ directly, and then by computing $P_{S\leftarrow T} \cdot [\mathbf{v}]_T$.

To find $P_{S\leftarrow T}$, put the vectors in as columns in the appropriate matrix and reduce: $[S \mid T] \rightsquigarrow [I \mid P_{S\leftarrow T}]$.

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \end{array}\right] \qquad P_{S\leftarrow T} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

To find $[\mathbf{v}]_S = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, *for* $\mathbf{v}_1 = [1\ \ 2]$, $\mathbf{v}_2 = [0\ \ 1]$ *we need to solve*

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2.$$

$$[1\ \ 3] = x_1[1\ \ 2] + x_2[0\ \ 1]$$

Without doing any computations you can see that $[1\ \ 3] = [1\ \ 2] + [0\ \ 1]$, so $[\mathbf{v}]_S = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

To find $[\mathbf{v}]_T = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, *for* $\mathbf{w}_1 = [1\ \ 1]$, $\mathbf{w}_2 = [2\ \ 3]$ *we need to solve*

$$\mathbf{v} = x_1\mathbf{w}_1 + x_2\mathbf{w}_2.$$

$$[1\ \ 3] = x_1[1\ \ 1] + x_2[2\ \ 3]$$

This leads to the following system of equations.

$$\left.\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 1 \\ x_1 & + & 3x_2 & = & 3 \end{array}\right\} \qquad \left.\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 1 \\ & & x_2 & = & 2 \end{array}\right\} \qquad \left.\begin{array}{rcrcr} x_1 & & & = & -3 \\ & & x_2 & = & 2 \end{array}\right\}, \qquad \text{so } [\mathbf{v}]_T = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

Finally, we need to check that $P_{S\leftarrow T} \cdot [\mathbf{v}]_T = [\mathbf{v}]_S$, *and this is indeed true.*

$$\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

2. (8 pts) Find the rank and nullity of the following matrix. $\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 2 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 3 & 2 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & -1 & -4 & 3 \\ 0 & -2 & -8 & 4 \\ 0 & -1 & -4 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

At this point of the row-reduction you can see that the rank is 3, so the nullity must be $4 - 3 = 1$.