MATH 240 QUIZ 7 NAME SOLUTIONS

Prof. J. Beachy Friday, 11/1/02 Score / 20

Let $L : \mathcal{P}_1 \to \mathcal{P}_2$ be defined by $L(p(t)) = tp(t) + p(0)$. Consider the ordered bases $S = \{t, 1\}$ and $S' = \{t+1, t-1\}$ for $\mathcal{P}_1$ and $T = \{t^2, t, 1\}$ and $T' = \{t^2+1, t-1, t+1\}$ for $\mathcal{P}_2$. Find the matrix representation of L, first with respect to S and T, and then with respect to S' and T', respectively. Find the dimension of the range of L, and find the dimension of the kernel of L.

A typical vector in $\mathcal{P}_1$ has the form $at + b$, and in these terms we have

$L(at+b) = t(at+b) + b = at^2 + bt + b$.

Since we can identify $at+b$ with the coordinate vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbf{R}^2$ and we can identify the polynomial $at^2 + bt + b$ in $\mathcal{P}_2$ with the coordinate vector $\begin{bmatrix} a \\ b \\ b \end{bmatrix}$ in $\mathbf{R}^3$, we can write the same thing in matrix form as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \\ b \end{bmatrix}.$$

This solutions to Homework 6 showed how to find the matrix relative to the other bases. We got

$$\left[\begin{array}{rrr|rr} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & -1 \end{array}\right] \rightsquigarrow \left[\begin{array}{rrr|rr} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & -2 \end{array}\right] \rightsquigarrow \left[\begin{array}{rrr|rr} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 2 & 1 & -3 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{rrr|rr} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{array}\right] \rightsquigarrow \left[\begin{array}{rrr|rr} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -3/2 \end{array}\right] \qquad M_{T' \leftarrow S'}(L) = \begin{bmatrix} 1 & 1 \\ 1/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$$

and so we have the matrix form

$$\begin{bmatrix} 1 & 1 \\ 1/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} c+d \\ 1/2c + 1/2d \\ 1/2c - 3/2d \end{bmatrix}.$$

This really represents this form of the function, written in terms of polynomials:

$$L(c(t+1) + d(t-1)) = (c+d)(t^2+1) + \left(\frac{c+d}{2}\right)(t-1) + \left(\frac{c-3d}{2}\right)(t+1).$$