EXAMPLE: Choosing a nonstandard basis to make it easier to find the matrix of a linear transformation.

Problem: In $\mathbf{R}^3$, find the matrix (relative to the standard basis) that describes a reflection in the plane $ax + by + cz = 0$.

Solution: Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be the reflection in the given plane. Let S be the standard basis. To try to make the calculations easier, we choose another basis tied to the plane. Let the first basis vector be the normal to the plane, and then choose two other independent vectors in the plane. (The given vectors seem to be easy to use, but to be sure they are independent we need to assume that $a \neq 0$. Of course, we will have to pay the price somewhere, and it will be in finding the transition matrices from one basis to the other.)

For $T = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} b \\ -a \\ 0 \end{bmatrix}, \begin{bmatrix} c \\ 0 \\ -a \end{bmatrix} \right\}$ the matrix $M_{T\leftarrow T}(L)$ of L with respect to T is easy to find.

The reflection L must take the normal vector into its negative, since a vector perpendicular to the plane is reflected directly across the plane. The other two basis vectors are left unchanged by L, since a vector already in the plane is its own reflection.

Let $M = M_{T\leftarrow T}(L) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P = P_{S\leftarrow T} = \begin{bmatrix} a & b & c \\ b & -a & 0 \\ c & 0 & -a \end{bmatrix}$. Then $P_{T\leftarrow S} = P^{-1}$, and inverting this matrix by row reduction looks like it will be pretty hard. Fortunately, there is another way: we can use the general formula $P^{-1} = \dfrac{1}{\det(P)}(\operatorname{adj} P)$, where $\operatorname{adj} P$ is the adjoint of P, defined in Section 5.4 via cofactors. (See page 342 of the text.) You can check that $\det(P) = a(a^2 + b^2 + c^2)$ and that $\operatorname{adj}(P) = \begin{bmatrix} a^2 & ab & ac \\ ab & -a^2 - c^2 & bc \\ ac & bc & -a^2 - b^2 \end{bmatrix}$. (Note that we happened to choose the basis in such a way that P is symmetric, so its inverse is also symmetric. This also shows why we assume that $a \neq 0$.)

Now $M_{S\leftarrow S}(L) = P_{S\leftarrow T} \cdot M_{T\leftarrow T}(L) \cdot P_{T\leftarrow S} = PMP^{-1}$

$$= \begin{bmatrix} a & b & c \\ b & -a & 0 \\ c & 0 & -a \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{\det(A)} \begin{bmatrix} a^2 & ab & ac \\ ab & -a^2 - c^2 & bc \\ ac & bc & -a^2 - b^2 \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} a & b & c \\ b & -a & 0 \\ c & 0 & -a \end{bmatrix} \begin{bmatrix} -a^2 & -ab & -ac \\ ab & -a^2 - c^2 & bc \\ ac & bc & -a^2 - b^2 \end{bmatrix}$$

$$= \frac{1}{a(a^2 + b^2 + c^2)} \begin{bmatrix} -a^3 + ab^2 + ac^2 & -2a^2b & -2a^2c \\ -2a^2b & -ab^2 + a^3 + ac^2 & -2abc \\ -2a^2c & -2abc & -ac^2 + a^3 - ab^2 \end{bmatrix}$$

$$= \frac{1}{a^2 + b^2 + c^2} \begin{bmatrix} -a^2 + b^2 + c^2 & -2ab & -2ac \\ -2ab & a^2 - b^2 + c^2 & -2bc \\ -2ac & -2bc & a^2 + b^2 - c^2 \end{bmatrix}$$

Example: Suppose that we want to reflect in the plane $y = x$. This should interchange the x and y coordinates, but leave the z coordinate fixed. Substituting $a = 1$ and $b = -1$ into the general answer does give the matrix we expect. (Whew!)

$$\frac{1}{2} \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$$